TRIGONOMETRY AND GRAPHS

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TRIGONOMETRY

FUNCTIONS OF ANGLES

GENERAL REMARKS

1. Parts of a Triangle.—A triangle may be considered as being made up of six parts, namely, three sides and three angles. The lengths of the sides are expressed in inches, feet, etc., and the angles in degrees and subdivisions of a degree. If three of the six parts of a triangle are known, one of them being the length of a side, the other three parts may be calculated. For example, if the lengths of the three sides are known, the sizes of the three angles can be found. If two angles and the length of one side are known, the other angle and the lengths of the two remaining sides can be calculated. If two sides and one angle are known, the other side and the two remaining angles can be found. The process of calculating the values of the unknown sides or angles, having given the values of three of the six parts of the triangle, is called the **solution of the triangle.**

2. Trigonometry.—There is one branch of mathematics that relates to the solution of triangles; accordingly, it is called **trigonometry.** This term is derived from two Greek words: *trigonon,* meaning *triangle;* and *metron,* meaning *measure.* Thus, trigonometry is simply the branch of mathematics that deals with the measurement and calculation of the parts of triangles.

3. Functions.—The work of calculating the values of the sides and angles of triangles is made fairly easy by the use of what are called *trigonometric functions of angles.* To understand this expression it is necessary, first of all, to know what is meant by a function. A **function** of any quantity is simply a second quantity, the value of which depends on the value of the first quantity. For example, the circumference of a circle always has a length equal to 3.1416 times the diameter; that is, the value of the circumference depends on the value of the diameter. Therefore, the circumference of a circle is a function of the diameter. On the other hand, the diameter of a circle is always equal to the circumference divided by 3.1416; hence, the diameter is a function of the circumference. The distance an electric car travels depends on the speed and the time; therefore, the distance traveled is a function of the time and speed. The resistance of an electric circuit is a function of the length of the circuit, because the resistance depends directly on the length.

4. Value of a Knowledge of Trigonometry.—A fair working knowledge of the principles of trigonometry is valuable to any one engaged in engineering work, but it is extremely important to the student of electrical engineering because it forms the basis for the solution of many problems in connection with alternating currents; in fact, the solution of many of these problems would be almost impossible for the average student if he were not able to use trigonometry. Besides, there are many other classes of problems to which it may be applied. For example, the power-plant man may use it to determine the height of a chimney. A foreman who is putting up wires may use it to find the height of a tree or of a standing pole. A workman fitting a piece of timber to act as a slanting brace may find by trigonometry the angles to which the ends of the piece must be cut. The variety of practical problems that may be solved by the principles of trigonometry is almost endless.

TRIGONOMETRIC FUNCTIONS

5. Functions of an Angle.—An angle—by which is meant the amount of opening between two straight lines that meet at a point—also possesses certain functions. To illustrate, take the case of the right triangle *A B C*, Fig. 1, which has a right angle at *C*. The sides of this triangle will be designated by *a*, *b*, and *c*, as shown, *a* being the side opposite the angle *A*, *b* that opposite the angle *B*, and *c* that opposite the right angle *C*. If the angle *A* were made larger, *a* would have to be longer; therefore, as the length *a* depends on the size of the angle *A*, it is a function of the angle *A*. At the same time, *c* would become longer, so that it also is a function of the angle *A*. Carrying this idea further, the size of the angle *B* likewise controls the relative lengths of the sides. It is impossible to change the size of an angle without causing changes in

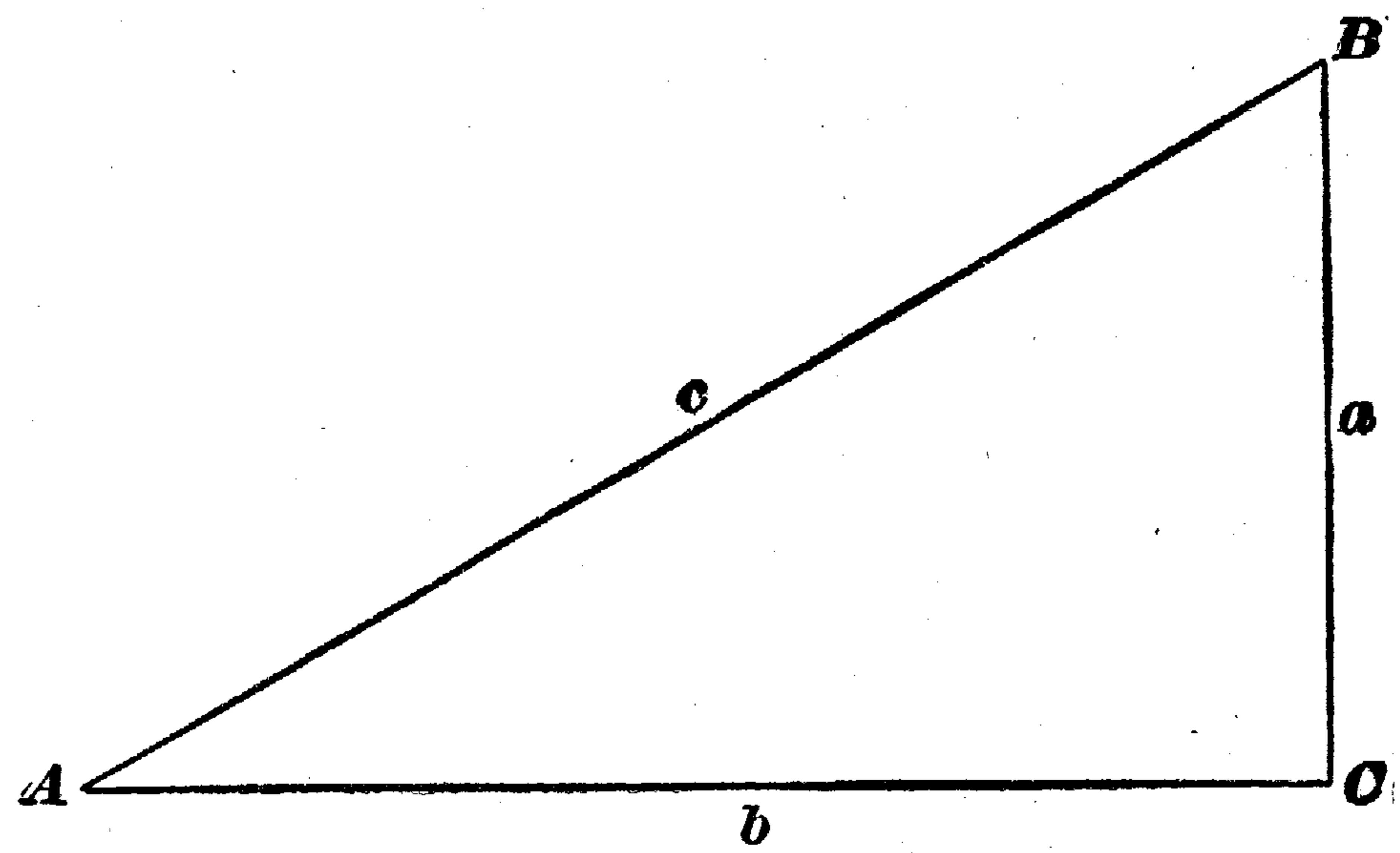


FIG. 1

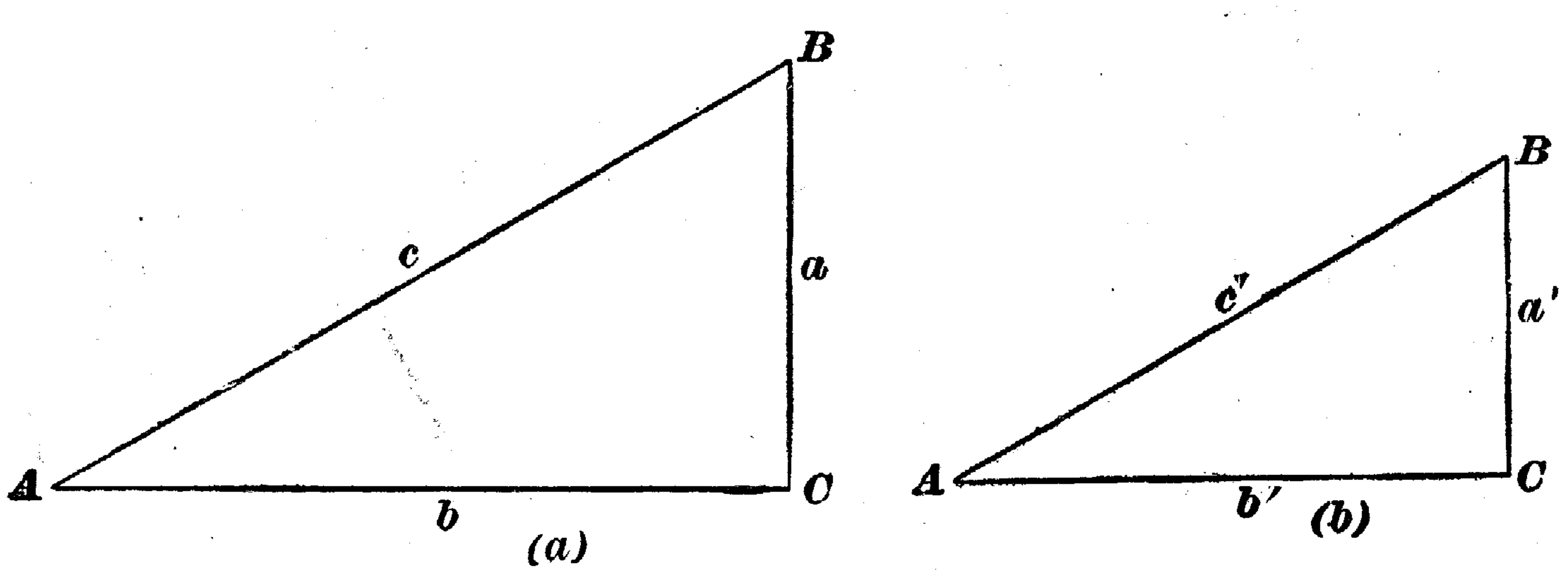


FIG. 2

the lengths of the sides, so long as *C* remains a right angle. Therefore, the sides of a right triangle are functions of the angles.

6. In Fig. 2 (*a*), suppose that the sides *b* and *c* make an angle of 30° with each other, or, in other words, that the angle A measures 30°; also, suppose that C is a right angle. Then, the side c opposite the right angle is the hypotenuse. Now, with reference to the angle A, the side a is the *side opposite* that angle and the side b is the *side adjacent* (or lying next) to that angle. If the hypotenuse c has a length of 2 inches, then, on measuring the side a, it will be found to be 1 inch long. The ratio of a to c will then be 1 : 2; that is, the ratio between the side opposite the angle A and the hypotenuse, when the angle A is 30°, has a value of $\frac{1}{2}$, or $\frac{a}{c}=\frac{1}{2}$.

7. Next, suppose that another right triangle is drawn, as in Fig. 2 (b), having an angle A of 30°, as before, but with a hypotenuse c' only $1\frac{1}{2}$ inches long. If the length of the side a' is measured, it will be found to be $\frac{3}{4}$ in., or just half as long as c'. Then, considering the 30° angle A, as before, the ratio between the side opposite and the hypotenuse is $\frac{3}{4} : 1\frac{1}{2}$, or $\frac{a'}{c'}=\frac{\frac{3}{4}}{1\frac{1}{2}}=\frac{1}{2}$. This value, it will be seen, is exactly the same as was obtained for the ratio between the side opposite and the hypotenuse in the case of the angle A in (a). Moreover, if other right triangles are drawn, of any size whatever, with one angle measuring 30°, the ratio between the side opposite that angle and the hypotenuse will always be 1 : 2, or have a value of $\frac{1}{2}$. This leads to the important conclusion that, so long as the angle A remains 30°, the side opposite will always be half as long as the hypotenuse; that is, the ratios $\frac{a}{c}$ and $\frac{a'}{c'}$ will always be $\frac{1}{2}$.

8. In the case of the triangles shown in Fig. 2, other ratios may be formed, besides the one just mentioned. For example, a ratio may be formed between the side a opposite the angle A and the side b adjacent to it. Furthermore, the values of this ratio for the two triangles in (a) and (b) are equal; that is, $\frac{a}{b}$ in (a) is equal to $\frac{a'}{b'}$ in (b), even though the lengths of a and b are different from the lengths of a' and b'. This statement may easily be proved, as follows: According to the principles of geometry, the sum of the three angles of every triangle is equal to 180°. In each of the two cases shown, the angle A is 30° and the angle C is a right angle, or 90°, the sum of the two, A and C, being 120°; therefore, the angle B in each must be $180° - 120° = 60°$. This means that the three angles of one are equal to the three angles of the other and so the two triangles are similar. Now, it is a law of geometry that if two figures are similar, their corresponding sides are proportional; so, $a : b = a' : b'$, or $\frac{a}{b} = \frac{a'}{b'}$. In this proportion, the first ratio refers to the sides of the triangle in (a) and the second ratio to the triangle in (b). This is the statement that was to be proved and shows that, so long as the angle A is unchanged in size, the ratio between the side opposite and the side adjacent is always the same, no matter how large or how small the right triangle may be.

9. In the preceding articles, only two different ratios have been considered—that of the side opposite and the hypotenuse and that of the side opposite and the side adjacent. There are several other ratios that may be formed. Every triangle has three sides, and therefore it is possible to form a ratio between one side and each of the other two remaining sides, or six ratios in all. In the triangle in Fig. 2 (a), these six ratios are $\frac{a}{b}$, $\frac{a}{c}$, $\frac{b}{a}$, $\frac{b}{c}$, $\frac{c}{a}$, and $\frac{c}{b}$. So long as the angles of the right triangle remain unchanged, these various ratios will not alter in value; and as they are formed by using the lengths of the sides a, b, and c, they are functions of the angles, because their values change when the sizes of the angles are changed.

10. Trigonometric Functions of Angles.—The six ratios stated in the preceding article form what are called the **trigonometric functions of an angle**, and each one is given a specific name so that it may be distinguished from the others. The ratio between the side opposite and the hypotenuse is called the **sine.** The ratio between the side adjacent and the hypotenuse is called the **cosine.** The ratio between the side opposite and the side adjacent is called the **tangent;** and the ratio between the side adjacent and the side opposite is called the **cotangent.** These are the functions most commonly used and are the only ones that will be referred to in the later parts of this Section. The two remaining ratios, or functions, are seldom used. They are: the **secant,** which is the ratio of the hypotenuse to the side adjacent; and the **cosecant,** which is the ratio of the hypotenuse to the side opposite.

11. When writing the names of the various ratios, it is customary to use abbreviations, so as to save time and space; thus, the word *sine* is abbreviated **sin,** *cosine* is abbreviated **cos,** *tangent* is abbreviated **tan,** and *cotangent* is abbreviated **cot.** Then, sin 26° is read *sine of 26°;* cos 15° is read *cosine of 15°;* tan 3° 27′ is read *tangent of 3° 27′.* It should be observed that these abbreviations are *not* followed by periods. It should also be observed that the word *of* is *not* used in connection with the abbreviations. Thus, although *sin* is the abbreviated form of *sine,* it stands for the two words *sine of,* in expressions like sin 21°, sin 12°, etc. Similarly *tan, cos,* and *cot* take the place of *tangent of, cosine of,* and *cotangent of,* although they are the abbreviations of the words *tangent, cosine,* and *cotangent,* respectively.

The ratios for the common functions of the angle *A,* Fig. 1, are as follows:

Ratio	Function	Abbreviation
$\frac{\text{Side opposite}}{\text{Hypotenuse}} = \frac{a}{c}$	sine of A	sin A
$\frac{\text{Side adjacent}}{\text{Hypotenuse}} = \frac{b}{c}$	cosine of A	cos A
$\frac{\text{Side opposite}}{\text{Side adjacent}} = \frac{a}{b}$	tangent of A	tan A
$\frac{\text{Side adjacent}}{\text{Side opposite}} = \frac{b}{a}$	cotangent of A	cot A

12. It should be noted carefully that the sine of the angle *A,* Fig. 1, is the ratio $\frac{a}{c}$ and that this particular ratio is correct only for sin *A.* If the angle *B* is considered, the side opposite the angle is *b,* the side adjacent is *a,* and the hypotenuse is c as before. Therefore, considering the angle B, the four main functions are as follows:

$$\sin B = \frac{b}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\cos B = \frac{a}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

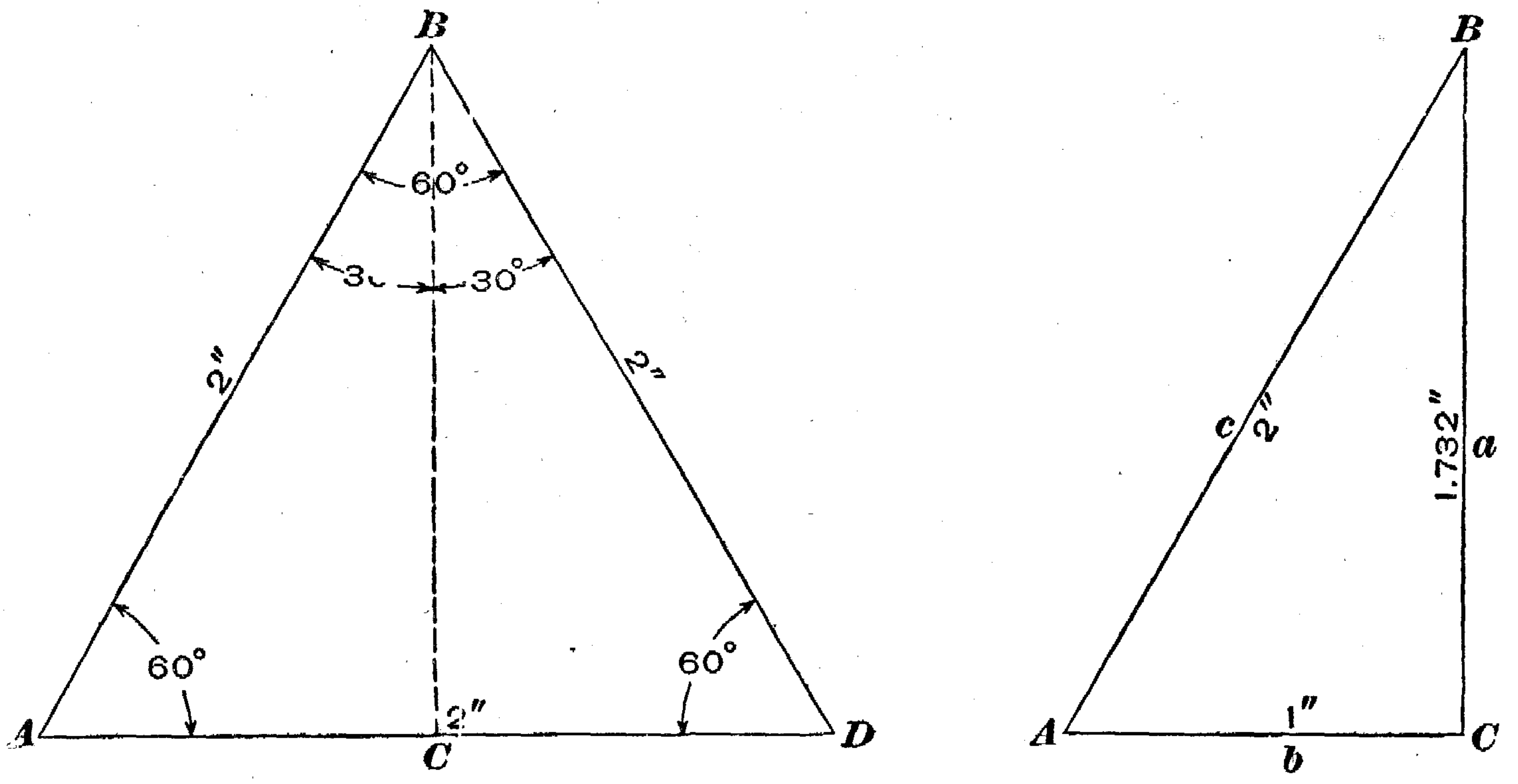
$$\tan B = \frac{b}{a} = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$\cot B = \frac{a}{b} = \frac{\text{side adjacent}}{\text{side opposite}}$$

The point to be kept in mind is that the sine is always the ratio between the side opposite and the hypotenuse; the cosine is always the ratio between the side adjacent and the hypotenuse; and so on for each of the other functions.

13. It will be observed that the ratios given in Arts. **11** and **12** are alike, but that they differ in their order and in their meanings. For example, Art. **11** shows that $\frac{a}{c} = \sin A$ and Art. **12** shows that $\frac{a}{c} = \cos B$. The ratio $\frac{a}{c}$ is the same in both cases and its value must be the same in both, because the sides a and c remain unchanged in length; therefore, it follows that $\sin A = \cos B$. Similarly, $\frac{b}{c}$ is equal to $\cos A$, but it is also equal to $\sin B$; therefore $\cos A = \sin B$. Again, $\frac{a}{b} = \tan A$ and $\frac{a}{b} = \cot B$; therefore, $\tan A = \cot B$. And finally, $\frac{b}{a} = \cot A$ and also $\frac{b}{a} = \tan B$; therefore, $\cot A = \tan B$. These relations are very important.

14. Deriving Functions of 30° and 60°.—An equilateral triangle, or a triangle with equal sides and equal angles, is shown in Fig. 3 (a). If a perpendicular line BC is drawn from B to the base AD, it will make two angles of 90° with the base and will divide the base into two equal parts. Suppose that the sides are 2 inches long; then, each part of the base is 1 inch long. The three angles of the triangle must be 60° each, because their sum must be 180°. The line $B\,C$ will divide the angle B into two equal parts, each 30°. Now, suppose that the left-hand triangle $A\,B\,C$ is set off by itself, as in (b), and



(a) FIG. 3 (b)

that the functions of the angles A and B are to be found. It is first necessary to find the length of the side a. As the triangle is a right triangle,

$$a=\sqrt{c^2-b^2}=\sqrt{2^2-1^2}=\sqrt{4-1}=\sqrt{3}=1.732 \text{ inches}$$

Considering the angle B, which is 30°, the side opposite is b and the side adjacent is a. Therefore,

$$\sin 30°=\sin B=\frac{b}{c}=\frac{1}{2}=.500$$

$$\cos 30°=\cos B=\frac{a}{c}=\frac{1.732}{2}=.866$$

$$\tan 30°=\tan B=\frac{b}{a}=\frac{1}{1.732}=.577$$

$$\cot 30°=\cot B=\frac{a}{b}=\frac{1.732}{1}=1.732$$

Considering the angle A, which is 60°, the side opposite is a and the side adjacent is b. Therefore,

$$\sin 60°=\sin A=\frac{a}{c}=\frac{1.732}{2}=.866$$

$$\cos 60° = \cos A = \frac{b}{c} = \frac{1}{2} = .500$$

$$\tan 60° = \tan A = \frac{a}{b} = \frac{1.732}{1} = 1.732$$

$$\cot 60° = \cot A = \frac{b}{a} = \frac{1}{1.732} = .577$$

15. Deriving Functions of 45°.—The functions of 30° and 60° are used more frequently than any others, particularly by electrical engineers, but the functions of 45° are also required very often. The triangle shown in Fig. 4 is a right triangle with each acute angle 45°. Let each side be 1 inch long; then, the length of the hypotenuse will be $\sqrt{1^2+1^2} = \sqrt{2} = 1.414$ inches. The side opposite and the side adjacent are the same for each angle *A* and *B*; therefore, the values of the functions of *A* are exactly the same as those of *B*. Thus, if the angle *A* is considered,

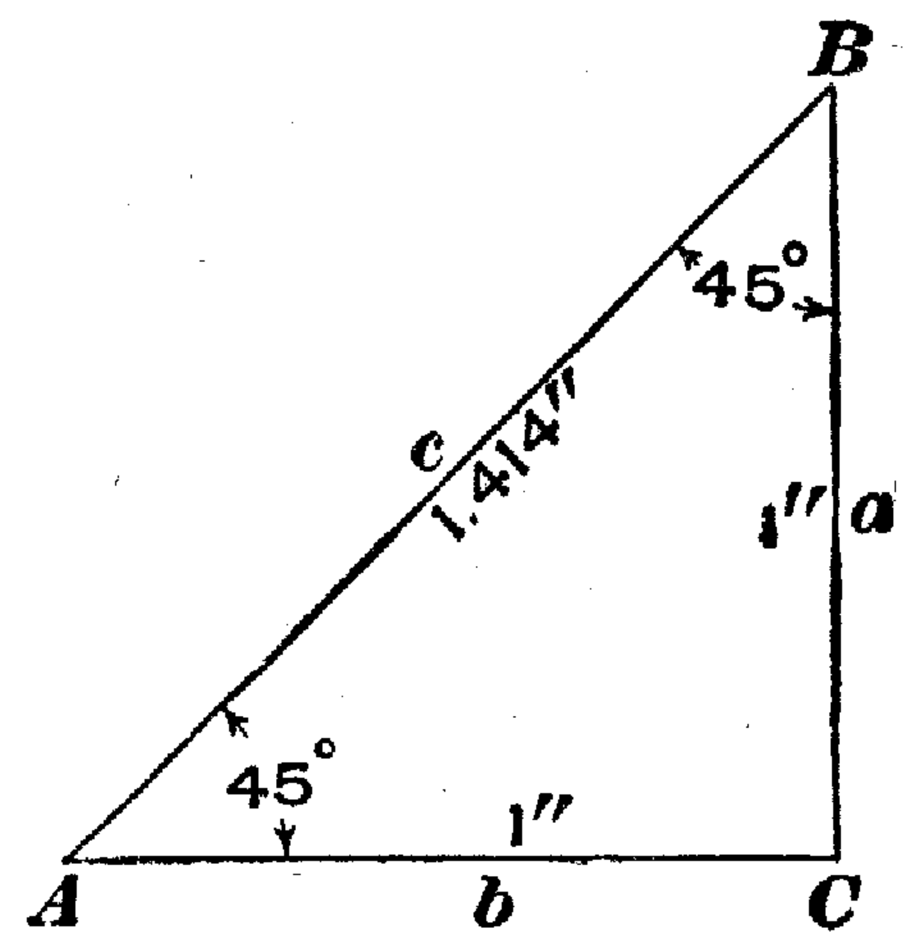


FIG. 4

$$\sin 45° = \sin A = \frac{1}{1.414} = .707$$

$$\cos 45° = \cos A = \frac{1}{1.414} = .707$$

$$\tan 45° = \tan A = \frac{1}{1} = 1.000$$

$$\cot 45° = \cot A = \frac{1}{1} = 1.000$$

Similarly, if the angle *B* is considered,

$$\sin 45° = \sin B = \frac{1}{1.414} = .707$$

$$\cos 45° = \cos B = \frac{1}{1.414} = .707$$

$$\tan 45° = \tan B = \frac{1}{1} = 1.000$$

$$\cot 45° = \cot B = \frac{1}{1} = 1.000$$

From the foregoing statements it may be seen that sin 45° = cos 45°, and therefore sin A = sin B = cos A = cos B. Similarly, tan 45° = cot 45° and therefore tan A = tan B = cot A = cot B. These relations are true, however, *only* when the angles A and B are equal and each is 45°.

16. Functions of Complementary Angles.—It has already been stated that the sum of the three angles of a triangle is 180°. As the angle C, Fig. 1, is 90°, the sum of the two remaining angles A and B must be 180° − 90° = 90°; that is, $A+B=90°$. This is true of every right triangle, no matter what its size or shape may be. In the particular case shown, A is 30° and B is 60°, their sum being 90°. Now, when any two angles have such values that, when added together, their sum is 90°, they are said to be *complementary angles*, and each is called the **complement** of the other. Thus, A and B are complements of each other, because their sum is 90°; so are 42° and 48°, 16° and 74°, 25° and 65°, and many others. If the complement of any angle is to be found, the given angle is subtracted from 90° and the remainder is the complement. Thus, the complement of 30° is 90° − 30° = 60°; the complement of 39° 45′ is 90° − 39° 45′ = 50° 15′; and the complement of 72° 12′ is 90° − 72° 12′ = 17° 48′.

17. As the angle A, Fig. 3 (b), is the complement of the angle B and the angle B is the complement of the angle A, the relations between their functions, as given in Art. **14**, may now be stated as follows:

$$\sin A = \cos B = \cos\,(90° - A)$$
$$\cos A = \sin B = \sin\,(90° - A)$$
$$\tan A = \cot B = \cot\,(90° - A)$$
$$\cot A = \tan B = \tan\,(90° - A)$$
$$\sin B = \cos A = \cos\,(90° - B)$$
$$\cos B = \sin A = \sin\,(90° - B)$$
$$\tan B = \cot A = \cot\,(90° - B)$$
$$\cot B = \tan A = \tan\,(90° - B)$$

These facts may be stated in a general way, as follows:

1. The sine of any angle is equal to the cosine of its complement.

2. The cosine of any angle is equal to the sine of its complement.

3. The tangent of any angle is equal to the cotangent of its complement.

4. The cotangent of any angle is equal to the tangent of its complement.

To illustrate these points clearly, suppose that, in Fig. 1, the sides *a*, *b*, and *c* have lengths of 1.25, 2.165, and 2.5 inches, respectively. Then,

$$\sin A = \cos B = \frac{a}{c} = \frac{1.25}{2.5} = .5 \qquad \tan A = \cot B = \frac{a}{b} = \frac{1.25}{2.165} = .577$$

$$\cos A = \sin B = \frac{b}{c} = \frac{2.165}{2.5} = .866 \qquad \cot A = \tan B = \frac{b}{a} = \frac{2.165}{1.25} = 1.732$$

18. Functions of Supplementary Angles.—If the sum of two angles is 180°, the angles are *supplementary* and each is said to be the **supplement** of the other; thus, 106° and 74° are supplementary angles, because 106°+74°=180°. For the same reason, 120° and 60° are supplementary, as are 89° and 91°, 27° and 153°, and many other combinations. To find the supplement of an angle, it is simply necessary to subtract that angle from 180°. Thus, the supplement of 52° 24′ is 180° − 52° 24′ = 127° 36′. The values of the trigonometric functions of two supplementary angles are numerically equal, each to each; that is, a function of an angle is numerically equal to the same function of its supplement. Therefore, considering any angle, as *A*, the following relations are true, so far as the numerical values of the functions are concerned:

$$\sin A = \sin\,(180° - A) \qquad \tan A = \tan\,(180° - A)$$

$$\cos A = \cos\,(180° - A) \qquad \cot A = \cot\,(180° - A)$$

Therefore, if the angle given is greater than 90° and less than 180°, and some function of it is required, subtract it from 180° and find the same function of the remaining angle.

19. Table of Natural Trigonometric Functions. From what has already been stated with regard to the trigonometric functions of angles, it is seen that for a given angle there is one unchanging value of its sine, a single value for its cosine, and so on for each of the other functions. It is therefore possible to arrange a table in which are given the various values of the functions corresponding to the different values of angles. Such a table will be found at the end of this Section. It gives the values of the sines, cosines, tangents, and cotangents of all angles from 0° to 90°, to five decimal places. These trigonometric functions are known as the natural functions of the angles, and so the values are called the natural sines, natural cosines, etc. This table is useful in solving many kinds of engineering problems, as will be shown later.

20. The table at the end of this Section is divided into two parts. The first nine pages of the table contain the values of the natural sines and cosines and the remaining nine pages contain the values of the natural tangents and cotangents. For all ordinary problems it will be found sufficiently accurate if the value of the angle is taken to the nearest minute. The table is therefore arranged so as to show the values of the functions for all angles from 0° to 90°, inclusive, increasing by 1′. All angles expressed in whole numbers of degrees from 0° to 44° will be found in the rows at the tops of the pages, the minutes being given in the first column of the page, reading downwards. The values of the functions are given in the columns headed by the names of the functions. For whole degrees from 45° to 89° it is necessary to look along the bottoms of the pages, the minutes being given in the right-hand column, reading *upwards*. As 90° is equal to 89° 60′, the values of the functions of 90° will be found on the first and tenth pages of the table, in the first line and in the second and third columns, as will be explained later.

It will be noticed that the columns headed Sine at the top are marked Cosine at the bottom, and those marked Cosine at the top are marked Sine at the bottom; similarly, the terms Tang and Cotang are interchanged at top and bottom of each column, in the latter half of the table. The reason for this will be understood from what is said in Art. **17.** The angle corresponding to any particular value in the table, reading the value of the angle from the *top,* is the complement of the angle corresponding to that value, reading the value of the angle from the *bottom*. That is why the table is read downwards from 0° to 45° and upwards from 45° to 90°.

21. The table gives the values of the functions of angles expressed in degrees and minutes only. So, if the value of an angle is given in degrees, minutes, and seconds, the angle should be taken to the nearest number of minutes and the function corresponding to this latter value should be found. As there are 60 seconds in 1 minute, the middle point is half of 60, or 30; so, if an angle contains less than 30″, ignore the number of seconds, but if it contains 30″ or more than 30″, drop the number of seconds and add 1′ to the number of minutes in the angle. These points may be illustrated by examples. For instance, consider the angle 12° 27′ 45″. As 45″ is greater than 30″, add 1′ to the number of minutes in the angle, which will give 12° 28′; that is, 12° 27′ 45″ is 12° 28′, approximately. Similarly, 65° 48′ 22″ is 65° 48′, approximately, as 22″ is less than 30″ and is ignored; 7° 18′ 32″ becomes 7° 19′, as 32″ is more than 30″; 47° 52′ 8″ is taken as 47° 52′; 29° 6′ 51″ is taken as 29° 7′; 35′ 40″ is taken as 36′; 4° 30′ 30″ is taken as 4° 31′; and so on.

22. Limits of the Natural Functions.—Between 0° and 90° the sine increases from a value of zero to a value of 1, while the cosine at the same time decreases from a value of 1 to a value of zero. The tangent of an angle of 0° is zero, from which value it increases as the angle increases, until its value for an angle of 90° is infinity. This word *infinity* simply means a value so great that it cannot be measured. The cotangent of an angle of 0° is infinity, and it decreases as the angle increases, until its value for an angle of 90° is zero.

23. Inverse Functions.—In various books and magazines will be found expressions like $A = \sin^{-1} .86603$, $B = \tan^{-1} .47382$, etc. These are trigonometric expressions and are called **inverse functions.** The first one is read *A equals the angle whose sine is .86603* and the second is read *B equals the angle whose tangent is .47382.* The only reason for using such an expression is that it saves writing a long phrase. The -1