FORMULAS

Serial 1982-2 Edition 2

ELEMENTARY OPERATIONS

USE OF SYMBOLS

RULES AND FORMULAS COMPARED

1. In the preceding Sections, and in particular in that one dealing with mensuration, frequent use is made of rules indicating how to use certain numbers for the purpose of finding an unknown length, area, or volume.

When the arithmetical processes are of a simple kind, the rule will be short and easily understood. If, however, the rule deals with several numbers that have to be combined and used in various ways, the rule becomes more complicated. It is then hard to see what must be done to apply the rule.

In order to simplify solutions a short method of expressing rules has been devised. In all arithmetical work a kind of shorthand is used. The sign + is written for the word plus; ÷ is used instead of divided by; cwt. is a symbol for 100 pounds or hundred-weight. Any one who reads a scientific or business magazine finds other shorthand expressions in which letters and the arithmetical symbols are used together. Such an expression is a short way of writing a rule, and it is called a *formula*.

A rule commonly used is: *The volume of a rectangular solid is equal to the product of its length, breadth and thickness.* This rule is often shortened by writing

$$volume = length \times breadth \times thickness$$

Suppose the first letter of each word is used as its abbreviation. Then,

$$v = l \times b \times t$$

If it is understood that when there is no sign between two letters they are to be multiplied together, it is possible to write the expression as

$$v = lbt$$

This is a shorthand way of expressing the rule for finding the volume of a rectangular solid, and it is a *formula.*

For a longer rule, there is more advantage in writing it the short way, but in these articles simple rules must be used as illustrations before the more complicated ones are considered.

EXAMPLE.—Express the following rule as a formula, using the first letter of a word as its abbreviation: The area of a triangle equals one-half the product of its base by its height.

SOLUTION.—First shorten the rule by using arithmetical symbols to indicate the operations:

$$\text{area} = \tfrac{1}{2} \times \text{base} \times \text{height}$$

Next abbreviate each word:

$$a = \tfrac{1}{2} \times b \times h$$

The multiplication signs may be omitted and the formula becomes

$$a = \tfrac{1}{2}\, b\, h$$

EXAMPLES FOR PRACTICE

Express each of the following rules as a formula. Use the first letter of a word as its abbreviation.

1. The area of a rectangle equals the product of the base by the height.
Ans. $a = b \times h$, or $a = b\,h$

2. The side of a square equals the square root of its area.
Ans. $s = \sqrt{a}$

3. The area of a circle equals the square of its radius multiplied by 3.1416.
Ans. $a = r^2 \times 3.1416$, or $a = 3.1416\, r^2$

4. To find the interest on a given principal multiply the principal by the rate for one year and then multiply the result by the time expressed in years.
Ans. $i = p \times r \times t$, or $i = p\, r\, t$

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2. Before formulas can be discussed, the general principles that govern letters used in them must be understood, just as the multiplication tables must be learned before it is possible to solve practical problems in which multiplication is used. In explaining the underlying principles, step by step, few practical applications can be shown, for a fair knowledge of all of the principles is necessary before their connection with formulas, which are used in the different trades and professions, can be understood.

ARITHMETICAL SIGNS AND THEIR APPLICATION

3. Signs of Aggregation.—The signs used in formulas are the ordinary signs indicative of arithmetical operations and the signs of aggregation. All these signs were explained in preceding Sections, but the signs of *aggregation* will need further explanation, as their application is extended in some directions.

The signs of **aggregation** are four in number, viz., —, ( ), [ ], and { }, respectively called the **vinculum,** the **parenthesis,** the **brackets,** and the **braces.** They are used to a great extent when it is necessary to indicate that all the numbers included by them are to be subjected to the same operation. For example, if the sum of 5 and 8 is to be multiplied by 7, any one of the four signs of aggregation may be used, but it is customary to use the *parenthesis;* thus, $(5+8)\times 7$. As already explained, the vinculum is used extensively in connection with the radical sign to indicate a root.

If *two* signs of aggregation are needed, the brackets and the parenthesis are used, so as to avoid having a parenthesis within a parenthesis, the brackets being placed outside. For example, $[(20-5)\div 3]\times 9$ means that the difference between 20 and 5 is to be divided by 3, and the result multiplied by 9.

If *three* signs of aggregation are required, the brace, brackets, and parenthesis are used, the brace being placed outside, the brackets next, and the parenthesis inside. For example, $\{[(20-5)\div 3]\times 9-21\}\div 8$ means that the quotient obtained by dividing the difference between 20 and 5 by 3 is to be multiplied by 9, and that after 21 has been subtracted from the product thus obtained, the result is to be divided by 8.

Many times a multiplication sign either preceding or following a symbol of aggregation is omitted; thus $(5+8)7$ or $7(5+8)$ means the same as $(5+8)\times 7$ or $7\times(5+8)$. In such cases the multiplication sign is understood but not written. Similarly $(2+3)\ (6+5)$ may be written instead of $(2+3)\times(6+5)$.

4. Order of Operations.—When several quantities are connected by the various signs indicating addition, subtraction, multiplication, and division, the operation indicated by the sign of *multiplication* must always be performed *first; next* in order comes the operation of *division*. Thus, $2+3\times 4$ equals 14, 3 being multiplied by 4 before adding it to 2. Similarly, $10\div 2\times 5$ equals 1, since 2×5 equals 10, and $10\div 10$ equals 1.

If this rule were not followed in the preceding examples, the results would be quite different. For instance, if in the example $2+3\times 4$ addition is performed before multiplication, as $2+3=5$, and $5\times 4=20$, the result differs from that previously found, which was 14.

In the example $10\div 2\times 5$, the quotient found by dividing 10 by 2 is 5, and 5 multiplied by 5 is 25. Performed in the correct manner the result was found to be 1.

5. In cases where several numbers are connected by using the sign of division and the plus or minus sign, the operation of division must be performed first. For example, $5-9\div 3$ is equal to 2; 9 divided by 3 gives 3 as the quotient, and $5-3=2$.

The signs of addition and subtraction are of equal value; that is, if several quantities are connected by plus and minus signs only, the indicated operations may be performed in the order in which the quantities are placed. Thus, in the example $5+7-4$, the solution may be found as $5+7=12$, and $12-4=8$. Or, $7-4=3$, and $3+5=8$.

EQUATIONS FORMED WITH NUMBERS

FUNDAMENTAL PRINCIPLES

6. Elementary Operations.—The principle on which any formula is based is a condition of equality existing between a combination of numbers arranged in a certain manner. The following example shows a simple combination of numbers of this kind:

$$2+4=5+1$$

The equality sign (=) is placed as the connecting link between the numbers 2 and 4 on one side, and the numbers 5 and 1 on the other side. An expression of this kind is known as an **equation,** because the equality sign connects two combinations of numbers that are equal, the sum of 2 and 4 being equal to the sum of 5 and 1.

This condition of equality is not limited to *sums* of numbers only, but may refer to any combination of numbers, subjected to the arithmetical operations of addition, subtraction, multiplication, division, square root, etc. The following shows an equation in which there is a combination of some of these operations:

$$(9\times 8)-2=7\sqrt{64}+\frac{42}{3}$$

It is found that if the operations are carried out as indicated, the results on both sides of the equality sign are equal to 70.

7. Quantity.—It is convenient to have a term that may be applied to the whole expression on either side of the equality sign. For this purpose the word *quantity* is in general use. The term **quantity,** as used in mathematics, is applied to any number or combination of numbers on which the ordinary arithmetical operations are to be performed. The numerical combinations $(9\times 8)-2$ and $7\sqrt{64}+\frac{42}{3}$, mentioned in the preceding example, may both be referred to as quantities.

When the term *quantity* is used with this meaning, an equation may be said to be an expression of equality between two quantities.

8. First and Second Members.—Distinction is made between the quantities on either side of the equality sign in an equation by applying the term *first member* to the quantity on the left-hand side of the equality sign. The quantity on the other side is called the *second member*. Speaking of both quantities, they are known as *the members of the equation*. In the following example, 5×9 is the first member and $60-15$ is the second member. Thus,

first member		*second member*
5×9	$=$	$60-15$

9. Comparison Between Equation and Scale.—Equations may be better understood by comparison with the ordinary scale, as used in stores. A scale of this kind, as shown in Fig. 1, has two pans *a* and *b*, which are supported by a lever, pivoted in the stand *c*. When the pointer *d* stands at the zero mark, it indicates that a condition of equality exists between the weights supported by the two pans; that is, the weights are in balance, or equilibrium. In the following examples the scale will be indicated diagrammatically by the balance lever, the two pans, and the weights they contain.

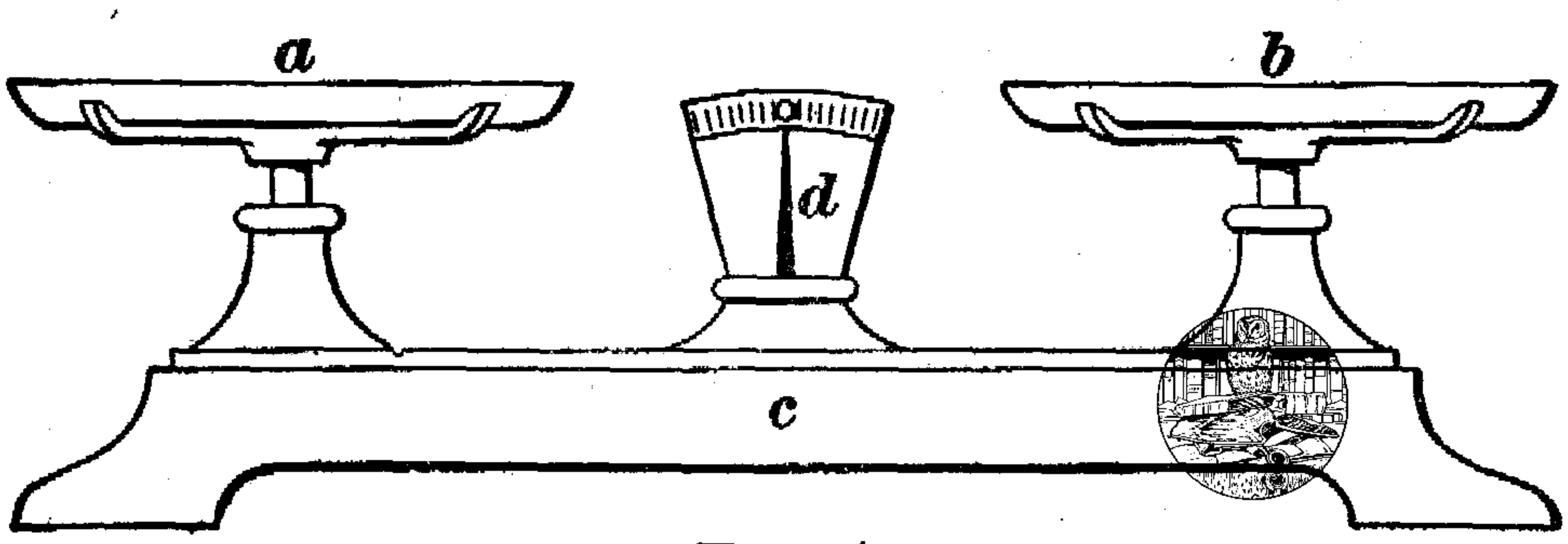

FIG. 1

10. A mathematical equation and a balance have this property in common, that a condition of equality cannot exist unless the quantities on both sides of the equality sign, or the weights in both pans of the scale, are equal.

It is impossible to reduce the weights placed in only one pan without disturbing the balance of the scale, just as it is impossible to reduce the value of only one member of the equation without disturbing the equality. But if the same change is made in both weights the scale will remain in balance, and likewise if the same change is made in both members of an equation they will remain equal.

The changes to which the members of an equation may be subjected are conveniently divided into the following four cases:

1. Adding the same quantity to both members of an equation.
2. Subtracting the same quantity from both members.
3. Multiplying both members by the same quantity.
4. Dividing both members by the same quantity.

11. Adding a Quantity to Both Members of an Equation.—It can be proved that when a number is added to one member of an equation, a number of equal value must be added to the other member if equality is to be maintained. For example, Fig. 2 (*a*) represents a scale with two pans *a* and *b*. The pan *a* supports weights of 10 and 8 pounds; and

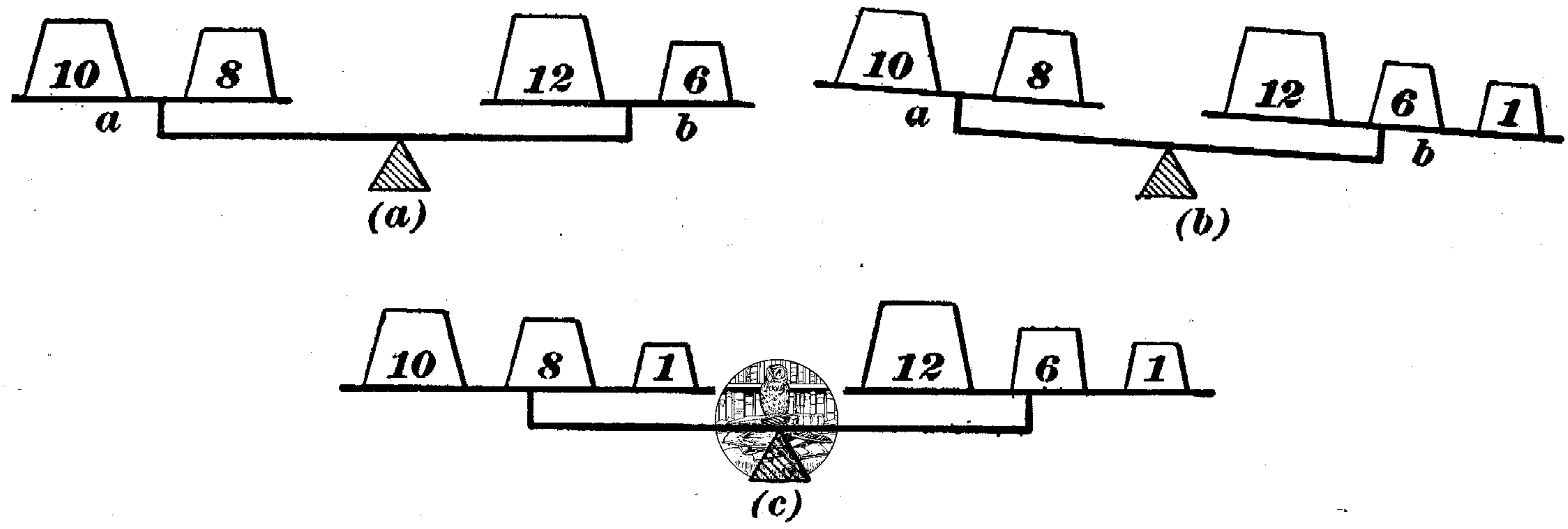

FIG. 2

the pan *b* supports weights of 12 and 6 pounds. The total weight on each pan is 18 pounds. If, now, 1 pound is added to the pan *b*, as in Fig. 2 (*b*), the balance is destroyed, but may be regained by placing a similar weight, 1 pound, on the pan *a*, as shown in Fig. 2 (*c*).

The arrangement shown in Fig. 2 (*a*) may be represented by an equation, as follows:

$$10+8=12+6$$

If 1 pound is added as in Fig. 2 (*b*), the resulting change may be shown in the following manner, the vertical dash (|) being used merely to separate the members on the two sides:

$$10+8 \mid 12+6+1$$

It is seen that the equality is destroyed and that this expression is not an equation, as the value 18 of the first member is not equal to the value 19 of the second member. To regain equality, a quantity equal to 1 may be added to the left member, making the two members equal, as follows:

$$10+8+1=12+6+1$$

It follows that if a quantity is added to one member of an equation, the same quantity must be added to the other member, to maintain the equality between the members.

12. Subtracting a Quantity From Both Members.—If a number is subtracted from one member of an equation, a number of the same value must be subtracted from the other member. For example, the weight on each of the scale pans *a* and *b*, Fig. 3 (*a*), is equal to 27 pounds. If the 2-pound

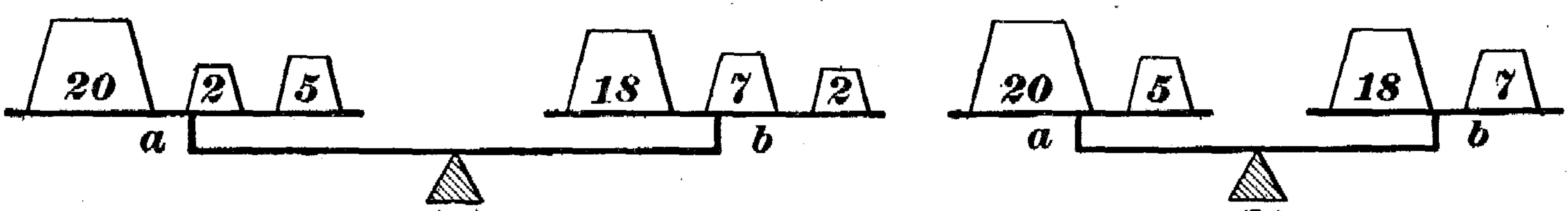

FIG. 3

weight is removed from the pan *a* the balance is destroyed, but it will be regained by removing the 2-pound weight from the pan *b*, as in Fig. 3 (*b*).

The arrangement shown in Fig. 3 (*a*) is represented by the following equation:

$$20+5+2=18+7+2$$

Removing the 2-pound weight from the pan *a*, Fig. 3 (*a*), is equivalent to subtracting 2 from the first member of the equation; thus,

$$20+5+2-2 \mid 18+7+2$$

Performing the subtraction indicated in the first member, the result is $2-2=0$, and the relation will assume the following form:

$$20+5 \mid 18+7+2$$

The equality is seen to be lost, and to regain it, the same value, 2, is subtracted from the second member, as follows:

$$20+5=18+7+2-2=18+7$$

From the preceding examples the following rule is derived:

Rule.—*If a quantity is added to or subtracted from one member of an equation, the same quantity must be added to or subtracted from the other member.*

if this rule is not complied with, the equality ceases to exist; that is, the quantity represented by one member is not equal to the quantity represented by the other member.

13. Multiplying or Dividing Both Members by Same Quantity.—The remarks made about adding a quantity to, or subtracting it from, one member of an equation apply also to the processes of multiplying or dividing a member by a given number. These operations are, in reality, operations of addition or subtraction, repeated a given number of times.

For example, let it be supposed that on the scale pan *a*, Fig. 4 (*a*), there is a weight of 15 pounds, and on the pan *b* the weights of 12 pounds and 3 pounds, also equal to 15

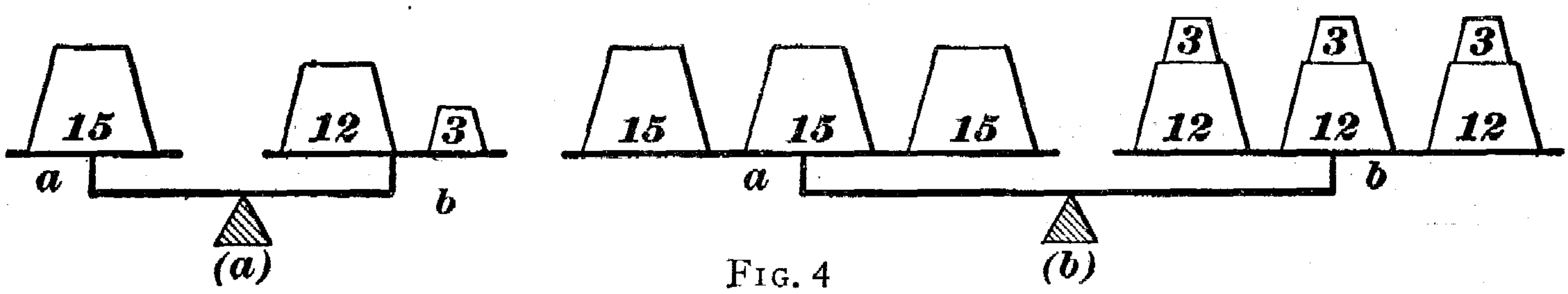

FIG. 4

pounds. It is required to multiply the weight on the pan *a* by 3. This is equivalent to increasing the total number of weights to three, or $15+15+15=45$ pounds. The condition will then correspond to that explained in Art. **11**; that is, a weight equal to $45-15=30$ pounds has been added to the pan *a*. If equilibrium is to be reestablished, the remedy is the same; that is, the weights on the pan *b* must be multiplied by the same number, or 3. These may be as shown in Fig. 4 (*b*), namely, three weights of 12 pounds each and three weights of 3 pounds each. The total weight on the pan *b* is $3\times12+3\times3=45$ pounds.

When this method is applied to an equation the same results are obtained. The arrangement shown in Fig. 4 (*a*) is represented by the equation

$$15=12+3$$

When the first member is multiplied by 3, the relation assumes the following form:

$$15\times3 \mid 12+3$$

This expression shows that the equality is lost, and to regain it, the second member must also be multiplied by 3, thus:

$$15\times3=(12+3)\times3=15\times3$$

The latter equation shows the application of the parenthesis, as explained in Art. **3**. In this case, it is necessary to consider the sum 12+3 as one number, when multiplying by 3. It is therefore enclosed in parenthesis.

14. Finally, it is necessary to consider the effect produced by dividing one member of an equation by a given number. In Fig. 5 (*a*) there is seen to be a balance between the weight of 18 pounds, resting on the pan *a*, and the two weights of 12 and 6 pounds, respectively, resting on the pan *b*.

If it is required to divide the weight of 18 pounds by 3, that is, to replace it with a weight one-third as great, or 6 pounds, the evident result will be that the scale is thrown out of

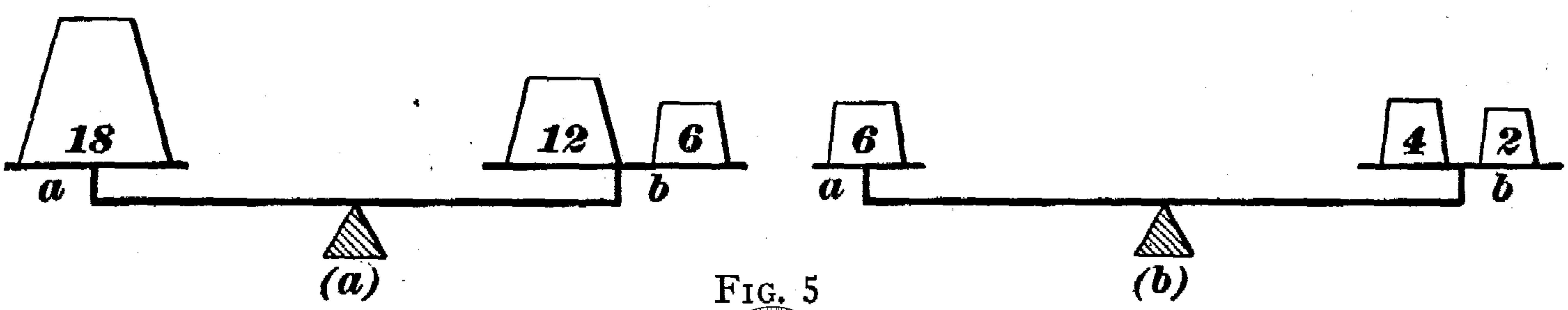

FIG. 5

balance. To regain equilibrium the weights on the pan *b* must be divided by the same factor, 3. One-third of 12 and of 6 pounds is 4 and 2 pounds, respectively. When the weights on the pan *b* are replaced by these reduced weights, the arrangement will appear as in Fig. 5 (*b*), where it is seen that the pans are again in equilibrium.

On the application of this principle to an equation, similar results are obtained. The arrangement shown in Fig. 5 (*a*) is represented by the equation

$$18 = 12 + 6$$

When the first member is divided by 3, the relation assumes the following form:

$$18 \div 3 \mid 12 + 6$$

or,

$$6 \mid 12 + 6$$

It is seen that, as 6 cannot be equal to 18, the equality is lost, and to regain it, the second member must also be divided by 3, as in the following equation:

$$6 = (12 + 6) \div 3 = 18 \div 3$$

or,

$$6 = 6$$

Also, in this equation, it is necessary to enclose the second member of the equation, or $12+6$, in a parenthesis, before dividing by 3. Otherwise, only the last number, 6, would be divided by 3.

15. Raising a number to a given power is a process of multiplication; that is, of multiplying a number by itself a given number of times. Also, extracting the root of a power is a process of division, as the root must go into its power, and the successive quotients, as many times as the index of the root indicates. Thus, a number may be divided three times by its cube root. For example, the cube root of 27 is 3; hence, $27 \div 3 = 9$, and $9 \div 3 = 3$, and $3 \div 3 = 1$.

It follows that the rules applying to the multiplication and division of the members of an equation by a given number also apply to raising the members to like powers or extracting like roots of both members. For example, in the equation

$$6 = 4 + 2$$

both members may be squared without affecting the equality of the equation. Thus,

$$6^2 = (4+2)^2$$

or,

$$36 = 36$$

Likewise may like roots of both members be extracted. For example, in the equation

$$64 = 44 + 20$$

the square root of both members may be extracted; thus,

$$\sqrt{64} = \sqrt{44+20}$$

or,

$$8 = 8$$

16. Transformation.—From the explanations given in Arts. **11** to **15** the following important rule is derived:

Rule.—*If one member of an equation is subjected to any arithmetical operation, the other member must be treated in the same manner.*

Changes made in an equation by applying this rule are called **transformations.**

This rule, which is apparently very simple, is of the greatest importance in operations with equations. If not complied with in all cases, serious errors will result. The application of the rule is shown by the following examples:

EXAMPLE 1.—It is required to add 25 to the second member of the equation

$$82+17+14=78+25+10$$

What changes must be made in the equation in order that the equality of the members may be maintained?

SOLUTION.—By the rule, it is necessary to add a number to the first member corresponding with that added to the second one. Or,

$$82+17+14+25=78+25+10+25. \quad \text{Ans.}$$

In its present form the value of each member is 138. In its original form the value was 113; but in both cases there exists a condition of equality.

EXAMPLE 2.—It is desired to subtract 34 from the first member of the following equation: $108+15+23=92+36+18$. What change must be made in the second member?

SOLUTION.—According to the rule, if 34 is subtracted from the first member, a number of equal value must be subtracted from the second member. The equation must, therefore, be written as follows:

$$108+15+23-34=92+36+18-34$$

On subtracting 34 from a larger number in each member, the equation will be as follows:

$$74+15+23=92+2+18. \quad \text{Ans.}$$

In its present form, each member of the equation is equal to 112. In the original equation the value was 146.

EXAMPLE 3.—If the second member in the equation $94+3=53+44$ is multiplied by 12, what changes must be made in the equation to maintain the equality of the members?

SOLUTION.—According to the rule, it is not allowable to multiply the second member by 12, unless the first member is multiplied by the same factor. The equation must, therefore, be written as follows:

$$(94+3)12=(53+44)12. \quad \text{Ans.}$$

Each member must be enclosed in a parenthesis to insure that both numbers are multiplied by 12, and not only the last one.

EXAMPLE 4.—The second member of the following equation is to be divided by 6. If the equality is to be maintained, what change is necessary in the equation?

$$81+17-20=64-4+18$$

SOLUTION.—According to the rule, the equation must be written as follows:

$$(81+17-20)\div 6=(64-4+18)\div 6. \quad \text{Ans.}$$

As the whole of each member is to be divided by 6, the members must be enclosed in a parenthesis as shown.

17. Transposition.—When equations are used, it is often necessary to change a number from one member to the other; changes of this kind are called **transpositions.**

$$15+9+3=20+6+1 \qquad (1)$$

Suppose that in equation **1** it is required to move the number 20 from the right side to the left side of the equality sign. It was explained in Art. **12** that the same number may be subtracted from both members of an equation. Performing this operation by subtracting 20 from both members, the equation will appear as follows:

$$15+9+3-20=20+6+1-20$$

But, in the second member, $20-20=0$. Hence, the equation will be

$$15+9+3-20=6+1 \qquad (2)$$

An examination of equation **2** shows that the equality of the equation is not affected, as, in the first member, $15+9+3-20=7$, and, in the second member, $6+1=7$.

When the equations **1** and **2** are compared,

$$15+9+3=20+6+1$$

and

$$15+9+3-20=6+1$$

it is seen that on transferring 20 from the second to the first member its sign is changed from + to −.

18. It remains to be shown how a number, preceded by a minus sign, may be transposed from one member to the other member of an equation. For example, in the equation

$$105+15-30=40+50$$

it is desired to transfer the number −30 from the first to the second member of the equation. According to the rule in Art. **16,** it is permissible to add the same number to both members. Hence, if 30 is added to both members, the equation will appear as follows:

$$105+15-30+30=40+50+30$$