POWERS AND ROOTS

Serial 1980 — Edition 1

INVOLUTION, OR FINDING POWERS OF NUMBERS

PERFECT AND IMPERFECT POWERS

DEFINITIONS AND RULES

1. Factors.—The **factors** of a number are those numbers which, when multiplied together, will equal that number. Thus, 5 and 3 are the factors of 15, since $5 \times 3 = 15$.

In engineering calculations it is often found necessary to multiply a number by itself one or more times; thus, $5 \times 5 \times 5 = 125$. In this case the number 125 consists of three equal factors, each of which is 5.

2. Powers.—A product obtained from several *equal* factors is called a **power** of the number that is used as the factor. The power is named according to the number of equal factors in the product. Thus, 9 is the **second power,** or **square,** of 3, as 9 is equal to the product of the *two* equal factors 3 and 3. A product is called the **third power,** or **cube,** of a number, if it contains *three* equal factors; thus, 64 is the third power of 4, as 64 is equal to the product of the three factors 4, 4, and 4, or $4 \times 4 \times 4 = 64$.

3. Squares and Cubes.—The term **square** is used for the second power of a number, because the area of a square is equal to the product of two equal numbers, each of which represents the length of one side. Thus, the area of a square is equal to the *second power* of a number that represents the length of one of its sides. For example: The side of a square is 4 feet; its area $=4\times4=16$ square feet.

The term **cube** is used for the third power of a number, because the volume of a cube is equal to the product found by using the length of one edge three times as a factor; that is, its volume is equal to the *third power* of a number representing the length of one of its edges. For example, the edge of a cube is 5 inches; its volume is $5\times5\times5=125$ cubic inches.

4. Involution.—The process of finding powers of quantities is called **involution.** The term *raise* is generally used in connection with this process. Thus, it is said that 7 is *raised* to the third power by using it as a factor three times, or $7\times7\times7=343$.

5. Exponents.—It is not sufficient to say that a *power* of a given number is to be found; one must also know *which* power is required, whether it is to be the second, the third, etc. For the purpose of indicating the required power of a number, a small number, called an **exponent,** is written to the right and near the top of the number. This number indicates the power to which a quantity is to be raised, or the number of times the quantity is to be used as a factor. Thus, in the expression 3^6, the number ^{6} is the exponent, and shows that 3 is to be used as a factor six times, or that 3^6 is a contraction of

$$3\times3\times3\times3\times3\times3$$

When an exponent is attached to a number it is read as in the following examples:

4×4 is written 4^2, and is read **four square,** or *four exponent two;*

$5\times5\times5$ is written 5^3, and is read **five cube,** or *five exponent three;*

$8\times8\times8\times8$ is written 8^4, and is read **eight to the fourth power,** or *eight exponent four.*

6. Use of Parenthesis.—When several numbers are connected by any of the arithmetical signs and the result is to be raised to a given power, the numbers must be written inside a parenthesis and the exponent outside it, as in the following example: $(2+5)^2=7^2=49$. Without the parenthesis the result would be: $2+5^2=2+25=27$.

The same rule applies if a fraction is to be raised to a given power. For instance, if the square of $\frac{3}{4}$ is to be found, it must be written $(\frac{3}{4})^2$. By omitting the parenthesis it will appear as if the exponent refers to the numerator alone, instead of applying to the numerator and the denominator. The effect produced by omitting the parenthesis may be seen from the following example: $\left(\frac{3}{4}\right)^2=\frac{3^2}{4^2}=\frac{9}{16}$. Omitting the parenthesis, the result is $\frac{3^2}{4}=\frac{9}{4}$.

7. Perfect and Imperfect Powers.—There are comparatively few numbers that can be separated into equal factors; these numbers are called **perfect powers.** Thus, 16 is a perfect power of 4, because $16=4\times4$; 216 is a perfect power of 6, because $216=6\times6\times6$. Numbers that cannot be separated into exactly equal factors are called **imperfect**

TABLE I

PERFECT SQUARES AND CUBES

n	n^2	n^3	n	n^2	n^3
1	1	1	6	36	216
2	4	8	7	49	343
3	9	27	8	64	512
4	16	64	9	81	729
5	25	125	10	100	1,000

powers. Thus, 10, 12, 15, and 20 are imperfect powers, because none of them is the product of equal factors. In the numbers from 1 to 1,000, inclusive, there are only 50 perfect powers, not counting 1, and of these only 30 are perfect squares and 9 perfect cubes.

Table I contains the squares and cubes of numbers from 1 to 10, inclusive. The column of numbers is headed by the letter *n*, which is an abbreviation of the term *number*. The squares and cubes of the numbers found in the first and fourth columns are given in the other columns, headed by n^2 and n^3, respectively. Thus, the square of 8 is 64 and the cube of 5 is 125.

8. Rule for Raising a Number to Any Power.—To find any power of a number, the following rule should be used:

Rule.—I. *To raise a whole number or a decimal to any power, use the number as a factor as many times as the power requires or as the exponent indicates.*

II. *To raise a fraction to any power, use the numerator and the denominator as factors as many times as the power requires or as the exponent indicates, and write these products as numerator and denominator, respectively.*

EXAMPLE 1.—What is the third power, or cube, of 35?

SOLUTION.—From Art. **3**, the expression *cube of a number* is equivalent to the number with 3 as an exponent. Applying the rule, $35^3=35\times35\times35$, or

$$\begin{array}{r} 35\\ 35\\ \hline 175\\ 105\\ \hline 1225\\ 35\\ \hline 6125\\ 3675\\ \hline \textit{cube}=42{,}875 \end{array}\quad \text{Ans.}$$

EXAMPLE 2.—What is the value of 1.2^2?

SOLUTION.—According to the rule, the number must be used as a factor the number of times indicated by the exponent, 2. Hence, $1.2^2=1.2\times1.2=1.44$. Ans.

EXAMPLE 3.—What is the value of $(\frac{5}{8})^2$?

SOLUTION.—The exponent is 2, so the numerator and the denominator must each be used twice as a factor. Then,

$$\left(\frac{5}{8}\right)^2=\frac{5^2}{8^2}=\frac{5\times5}{8\times8}=\frac{25}{64}.\quad \text{Ans.}$$

EXAMPLE 4.—What is the fourth power of 15?

Solution.—From Art. **5**, the fourth power of 15 is equivalent to 15^4. Applying the rule, $15^4 = 15\times15\times15\times15$, or

$$\begin{array}{r}
15\\
15\\
\hline
75\\
15\\
\hline
225\\
15\\
\hline
1125\\
225\\
\hline
3375\\
15\\
\hline
16875\\
3375\\
\hline
\textit{fourth power} = 50625
\end{array}\quad \text{Ans.}$$

Example 5.—What is the cube of .12?

Solution.—The cube is found by using the number three times as a factor; therefore, the cube of .12 is

$$.12\times.12\times.12 = .001728. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Raise the following to the powers indicated:

			Ans.	
(*a*)	85^2.		(*a*)	7,225
(*b*)	$\left(\frac{12}{13}\right)^2$.		(*b*)	$\frac{144}{169}$
(*c*)	6.5^2.		(*c*)	42.25
(*d*)	14^4.		(*d*)	38,416
(*e*)	$\left(\frac{3}{4}\right)^3$.		(*e*)	$\frac{27}{64}$
(*f*)	$\left(\frac{5}{6}\right)^3$.		(*f*)	$\frac{125}{216}$
(*g*)	$\left(\frac{7}{2}\right)^3$.		(*g*)	$\frac{343}{8}$
(*h*)	1.4^5.		(*h*)	5.37824

EVOLUTION, OR FINDING ROOTS OF NUMBERS

SQUARE ROOT

DEFINITIONS

9. Roots of Numbers.—It was stated in Art. **2** that the product obtained from a number of equal factors is called a power of the number. If the process is reversed and the number of equal factors into which a number may be separated is found, then any one of these factors is known as the **root** of the number. For instance, if the number 27 is divided into the three factors 3, 3, and 3, then any one of these factors is known as a root of this number, as $27=3\times3\times3$. This process of finding a root of a number is known as **evolution;** it is the reverse of *involution*. The term *extract* is generally used in connection with this process, and it is said that the root of the number is *extracted*.

10. Classification of Roots.—If a number is separated into *two* equal factors, one of these factors is known as the **square root** of the number. Thus, if 25 is separated into two equal factors 5 and 5, then 5 is the square root of 25, because $5\times5=25$. The square root of 49 is 7, because $7\times7=49$; the square root of 1.21 is 1.1, because $1.1\times1.1=1.21$.

If a number is separated into *three* equal factors, one of the factors is known as the **cube root** of the number. Thus, 3 is the cube root of 27, since $3\times3\times3=27$.

The **fourth root** of a number is one of the *four* equal factors into which the number may be separated. Thus, the fourth root of 256 is 4, because $4\times4\times4\times4=256$.

The **fifth root** of a number is one of the *five* equal factors into which the number may be separated. Thus, 7 is the fifth root of 16,807, since $7\times7\times7\times7\times7=16{,}807$.

11. Derivation of Terms Square Root and Cube Root.—The terms *square* and *cube,* when applied to roots, are derived from the same source as similar terms applied to powers of numbers. Reversing the conditions stated in Art. **3,** it follows that, if a number represents the area of a square, the *square root* of the number must be equal to the length of one side of the square. For instance, if the area of a square floor is 81 square yards, the square root of 81, or 9, represents, in yards, the length of one side of the room.

If a given number represents the contents of a cube, the *cube root* of the number must give the length of one of the edges. Thus, if the contents of a cubical box is 27 cubic feet, the cube root of the number, or 3, gives the length of one of the edges, in feet.

12. Radical Sign.—The fact that the root of a number is to be extracted is usually indicated by placing the *radical sign* $\sqrt{}$ in front of it. The term **radical** is derived from the Latin word *radix,* meaning *root.* A *vinculum* $\overline{\quad}$ is connected with the radical sign and placed over the quantity to which the radical sign applies. Thus, $\sqrt{4,574,300}$ indicates that the root of the number following the radical sign is to be extracted.

13. Index.—Although the radical sign shows that a root is to be extracted, it must also show what root is required, and for this purpose an **index** is used. This is a small figure placed *above* the radical sign; thus, $\sqrt[2]{100}$ indicates the *square root* of 100, and $\sqrt[3]{1,728}$ indicates the *cube root* of 1,728. However, when the square root of a number is to be extracted, the index is usually omitted. Thus, $\sqrt{81}$ means the square root of 81; also, $\sqrt{22.48}$ means the square root of 22.48.

CALCULATION OF SQUARE ROOT

SQUARE ROOTS OF WHOLE NUMBERS AND OF DECIMALS

14. Introduction.—In calculating, or extracting, the square root of a number, it is necessary to perform a number of separate operations in a certain order. If the nature of each operation is kept in mind and if the operations are performed successively in the correct order, the process of extracting a square root should not offer any particular difficulties.

To facilitate the first part of the calculation, it is well to memorize the squares of the first twelve integers, or whole numbers, given herewith. The first line gives the numbers and the second line the corresponding squares.

Integers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
Squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144

15. Example of Extracting Square Root.—The method of finding the square root can best be explained by using an actual example and describing each step of the work. The first step is to point off, or separate, the number into periods, or parts, each containing two figures. Thus, suppose the square root of the whole number 31,505,769 is to be extracted. Ignore the commas that are used to divide the number and write it without them, thus: 31505769. Now, beginning at the right-hand figure, point off, or separate, the number into periods of two figures each, proceeding toward the left and using the mark ′ to separate the periods. The number will then appear as follows: 31′50′57′69. In case the whole number contains an odd number of figures, there will be only *one* figure in the period at the left. For instance, take the number 53,361. When this is pointed off it becomes 5′33′61, in which there are three periods. Here the 5 forms the first period, even though it consists of only one figure.

The reason for thus pointing off the number is to find out how many figures there will be in the root of the number. It is always true that the number of figures in the root is equal to the number of periods into which the number is divided. Thus, as 31′50′57′69 contains four periods, it is known at once, before any calculations are made, that there will be four figures in the square root of that number. Similarly, as 5′33′61 has three periods, the square root must contain three figures; and this is the case, as the square root of 53,361 is 231. After the number has been properly pointed off, the square root is extracted in the manner shown in the following example:

EXAMPLE.—Extract the square root of 31,505,769.

SOLUTION.—

	steps	*number* *root*	
(*a*)	5	31′50′57′69(5613	
	20	$5^2=25$	
first trial divisor	100	650	*first dividend*
	6	636	
first complete divisor	106	1457	*second dividend*
(*b*)	56	1121	
	20	33669	*third dividend*
second trial divisor	1120	33669	
	1		
second complete divisor	1121		
(*c*)	561		
	20		
third trial divisor	11220		
	3		
third complete divisor	11223		

EXPLANATION.—Beginning at the right, the number 31,505,769 is pointed off into periods of two figures each, as already explained. The largest single number whose square is less than 31, the first period, is now found. This is evidently 5, since the square of 6, or 36, is greater than 31. This number, 5, is written to the right of the number, as in long division; it is also written to the left, as at (*a*). The square of this first figure of the root, or $5^2=25$, is written under the first period, as shown, and is subtracted from it, leaving 6 as a remainder. The second period of the number is annexed to this remainder, giving 650 as the **first dividend.**

The first figure of the root, written at (*a*), is now multiplied by 20, giving a product 100, which is called the **first trial divisor.** The first dividend, 650, is now divided by this first trial divisor, 100, and the quotient 6 is obtained, which is *probably* the second figure of the root. This figure is written in the root, as shown, and is also added to 100, the first trial divisor, giving the sum 106, which is called the **first complete divisor.**

The first complete divisor, 106, is multiplied by 6, the second figure in the root, giving the product 636, which is subtracted from the first dividend; the remainder is 14, to which the two figures, 57, in the third period of the number are annexed, giving 1457 as the **second dividend.** The two figures of the root, 56, are now written at (*b*) and multiplied by 20, thus giving 1120, which is the **second trial divisor.** Dividing 1457 by the second trial divisor, 1120, the quotient 1 is obtained as the third figure of the root. Adding this figure to the second trial divisor, the result is 1121, which is the **second complete divisor.** Multiplying this divisor by 1, the third figure in the root, gives the product 1121, which is written under the second dividend, 1457. Then, subtracting it from the second dividend, the remainder is 336, to which the fourth period, 69, of the number is annexed, giving 33669 as the **third dividend.**

The three figures 561 in the root are now placed at (*c*) and multiplied by 20, giving the product 11220, which is the **third trial divisor.** Dividing 33669 by the third trial divisor, 11220, the quotient, 3, is obtained as the fourth figure of the root. Adding this figure to 11220. the result is 11223, the **third complete divisor.** Multiplying this divisor by 3, the fourth figure of the root, the product is 33669, which is written under the third dividend and subtracted from it, leaving no remainder. It follows that $\sqrt{31{,}505{,}769}=5{,}613$, and that $31{,}505{,}769=5{,}613 \times 5{,}613$.

16. Choosing Figures for the Root.—In the preceding explanation, where reference was made to finding the second figure, 6, of the root, the statement was made that *probably* the figure 6 would be the one required. The word *probably* is used here, because the various figures that are successively selected for the root are, at first, only trial figures. There can be no certainty that each of these figures will be the correct one, and not too large, until it is multiplied by the corresponding complete divisor, thus allowing the resulting product to be compared with the dividend. Referring to the foregoing example, it is not certain that 6, the second figure in the root, is correct, before it has been multiplied by the first complete divisor, 106, and the product found to be not greater than the first dividend, 650. In this case the product 636 is less than 650; but, suppose that the first dividend were 620, as in the example that follows. The product 636 would be too large, thus indicating that the figure 6 in the root is too large. It would then be necessary to substitute, in place of the 6, the figure 5 as the second figure in the root. The remaining part of the solution is simply a repetition of that already described.

			root
(*a*)	5	31'20'33'96 (5586	
	20	$5^2 = 25$	
first trial divisor	100	620	*first dividend*
	5	525	
first complete divisor	105	9533	*second dividend*
(*b*)	55	8864	
	20	66996	*third dividend*
second trial divisor	1100	66996	
	8		
second complete divisor	1108		
(*c*)	558		
	20		
third trial divisor	11160		
	6		
third complete divisor	11166		

17. Main Features of Extracting Square Root.—In order that the main features of calculating a square root may be remembered, it should be noted that the various trial divisors are obtained by multiplying the existing figures in the root by 20. Thus, the *first* figure in the root multiplied by 20 gives the *first* trial divisor; the *first two* figures of the root multiplied by 20 gives the *second* trial divisor, and so forth. Considering this feature as the framework of the process, the remainder will easily be remembered.

18. Method of Procedure When Trial Divisor is Larger Than Its Dividend.—Sometimes, in extracting a root, it may happen that a trial divisor is larger than the corresponding dividend. In such a case the method adopted is similar to that used in long division; that is, a cipher is annexed to the root and to the trial divisor and the next period is annexed to the last remainder to form a new dividend, larger than the new trial divisor. The following example shows how to proceed:

EXAMPLE.—Find the square root of 255,025.

SOLUTION.—

			root	
(*a*)	5	25'50'25(505		
	20	$5^2=25$		
first trial divisor	100	5025	*second dividend*	
second trial divisor	1000	5025		
	5			
second complete divisor	1005			

EXPLANATION.—After the number is separated into periods, it is seen that the root of the first period must be 5. This figure is written to the right of the given number and also at (*a*). The square of 5, or 25, is subtracted from the first period, leaving no remainder. The second period, 50, is brought down as the first dividend and the number 5 at (*a*) is multiplied by 20, giving 100 as the first trial divisor. As this trial divisor, 100, is not contained even once in the first dividend, 50, a cipher is annexed to the root and to the divisor, making the root 50 and the new, or second, trial divisor 1000. The third period of the number, 25, is now brought down and annexed to the remainder 50, making 5025, which is the second dividend. Dividing by the second divisor, 1000, the quotient, 5, is placed as the last figure of the root. Adding 5 to 1000 gives 1005 as the second complete divisor. Multiplying by 5 gives 5025, which is equal to the second dividend; hence, there will be no remainder.

In ordinary practice, instead of writing a separate second trial divisor, as in this example, the cipher would be annexed directly to the first trial divisor. Thus, a cipher could have been annexed to the first trial divisor, 100, giving the second trial divisor, 1000.

19. Extracting Square Roots of Decimals.—The square of any number, wholly decimal, always contains twice as many decimal places as the number squared. For example, $.1^2=.01$, $.13^2=.0169$, $.751^2=.564001$, etc. Conversely, it follows that the square root of a decimal contains only one half the number of decimal places found in the decimal itself. Thus, $\sqrt{12.723489}=3.567$. If the decimal contains an uneven number of figures, a cipher must be annexed to give an even number, as will now be explained.

When the square root of a decimal is to be extracted, the decimal is pointed off into periods of two figures each, *begin-*

ning at the decimal point and going to the *right.* Then, if the last period contains but one figure, a cipher is annexed to complete the period. Thus, the decimal .62371 is pointed off as follows: .62′37′10. Annexing ciphers to the right of a decimal does not change its value, as was explained in a preceding Section.

If the decimal is a portion of a mixed number, as 142.716, the whole-number part is pointed off to the *left* and the decimal part to the *right,* beginning at the decimal point in both cases, and considering the decimal point as a mark of separation. Thus, the number 142.716 is pointed off as follows: 1′42.71′60.

The operation of finding the square root in all these cases is similar to that previously described, except that when the decimal point is reached, a decimal point is placed in the answer. There will be as many decimal places in the root as there are periods in the decimal part of the number.

20. The following examples will show the method of extracting roots of decimals:

EXAMPLE 1.—Find the square root of 606.6369.

SOLUTION.—

			root	
	2		6′06.63′69(24.63	
	20	$2^2=$	4	
first trial divisor	40		206	*first dividend*
	4		176	
first complete divisor	44		3063	*second dividend*
			2916	
	24			
	20		14769	*third dividend*
			14769	
second trial divisor	480			
	6			
second complete divisor	486			
	246			
	20			
third trial divisor	4920			
	3			
third complete divisor	4923			

EXAMPLE 2.—What is the square root of .000576?

SOLUTION.—

		root
	2	.00'05'76(.024
	20	2^2= 4
trial divisor	40	176
	4	176
complete divisor	44	

EXPLANATION.—Beginning at the decimal point and separating the decimal into periods of two figures each, it is seen that the first period is composed of ciphers; hence, the first figure in the root must be a cipher. The decimal is now treated as if 5 were the first figure and the calculation is continued in the manner previously explained.

21. Perfect and Imperfect Powers.—Comparatively few numbers are exact squares; consequently, it is only in a small number of cases that the exact square root can be found. A number that has an exact root is called a **perfect power;** in the case of an exact square root the number from which it is extracted is termed a **perfect square.** The factors of the perfect powers are called **rational factors.** An exact square root of a number represents one of the two equal rational factors into which it is possible to separate the perfect square. For instance, $\sqrt{81}=9$, and $9\times9=81$; hence, the two numbers 9 are rational factors.

Numbers that cannot be separated into exactly equal factors are called **imperfect powers,** and the factors are called **irrational factors.** Any number, that cannot be divided into as many rational factors as there are units in the index of the root, will have a root with an unending decimal. For example, 20 lies between 16 $(=4^2)$ and 25 $(=5^2)$; hence, the square root of 20, or $\sqrt{20}$, is greater than 4 and less than 5, and is therefore equal to 4 plus an unending decimal. In other words, no matter to how many figures the square root of 20 may be calculated, the root will never be found exactly. Numbers ending in 2, 3, 7, or 8 are imperfect squares.

22. Extracting Root of Imperfect Power.—Although the square root of an imperfect power cannot be found exactly, as close an approximation may be obtained as is desired. The root may be carried to any required number of decimal places by annexing periods of two ciphers each to the number. In practice, five significant figures are all that are likely to be required, and four are generally sufficient.

EXAMPLE 1.—What is the square root of 3? Find the result to five decimal places.

	1		*root*
	20		3.00′00′00′00′00 (1.73205+
	20	$1^2=1$	
	7		200 *first dividend*
first divisor	27		189
			1100 *second dividend*
	17		1029
	20		
	340		7100 *third dividend*
	3		6924
second divisor	343		1760000 *fifth dividend*
			1732025
	173		27975
	20		
	3460		
	2		
third divisor	3462		
	1732		
	20		
	346400		
	5		
fifth divisor	346405		

EXPLANATION.—As five decimal places are required, it is necessary to annex five periods of ciphers to the right of the decimal point. The method employed in extracting the square root is the same as explained in the preceding examples. Attention is called only to the omission of the fourth divisor and dividend. After the fourth period is annexed to the remainder, 176, making the fourth dividend 17600, it is found that this dividend does not contain the divisor 34640. A cipher is therefore placed as the next figure in the root and a cipher annexed to the divisor, changing it into a fifth trial divisor. Another period of two ciphers is now annexed to the fourth dividend, changing it into a fifth dividend, and the calculation is continued, 5 being obtained as the next figure of the root. The required five decimal places have now been obtained and the operation could stop at this point; but in the