RATIO AND PROPORTION

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RATIO

SIMPLE RATIO

EXPRESSING AND FINDING RATIOS

1. Comparison of Numbers.—In practice there are many cases in which it is more useful to know how many times one number is larger than another number, than to know the actual values of these numbers. For example, in calculating the relative speeds of shafts driven by gear wheels, it is more convenient to state how many times more teeth one wheel has than another, than to consider the actual number of teeth on each wheel. These comparative values of the number of teeth are generally stated in the form of a fraction, as $\frac{3}{1}$, the fraction meaning that the number of teeth on the driving wheel is three times as great as that on the driven wheel. The actual number of teeth on the two wheels is left out of consideration for the time being; they may be 90 and 30, 36 and 12, or any other combination in which one wheel has three times as many teeth as the other.

2. Another example showing the advantage of giving the relative values of numbers may be found in government health reports. One of those may state, for instance, that of each 1,000 inhabitants in a certain city 25 persons suffered from influenza, 1.24 from typhoid, etc. Reports given in this form are much more instructive than those which simply state that in the same city 1,250 persons suffered from influenza, 62 from typhoid, etc. By means of the first report it is possible to institute comparisons with other cities, but with reports of the latter kind comparisons are difficult, unless the total number of inhabitants is known in each case.

Similarly, when speed of operation or production is considered, it is much easier to make comparison between various machines, when their relative speeds are known. For example, on comparing the speeds of two railway trains, it is found that a train A makes 90 miles in 2 hours 15 minutes and that another train B makes 108 miles in 2 hours 24 minutes. From those statements it is difficult quickly to form an idea of the relative speeds of these trains. But, if it is said that the train A made 40 miles an hour and the train B 45 miles an hour, the comparison is easier, and is still more facilitated if the relative speeds are stated in the form of a fraction, as for instance, the speed of A is $\frac{8}{9}$ of the speed of B. This expression means that while the train A makes 8 miles, the train B makes 9 miles.

It is the purpose of this Section to show by what means the relative values of two numbers may be found, and how the existing relation between one pair of numbers may be used for finding another pair, similarly related. This Section deals with one of the most interesting and useful subjects to be found in arithmetic.

3. Finding the Ratio of Two Numbers.—Two numbers may be compared in one of two ways. Suppose, for instance, that the diameters of two belt pulleys are to be compared, one being 30 inches and the other 6 inches. If it is necessary to know how many times 30 is larger than 6, then 30 is divided by 6 giving 5 as the quotient; thus, $30 \div 6 = 5$. Hence, it may be said that 30 inches is 5 times as large as 6 inches, or that the 30-inch diameter contains 5 times as many inches as the 6-inch diameter. Or, the numbers 30 and 6 may be compared by ascertaining what part 6 inches is of 30 inches

Then, 6 is divided by 30, giving the quotient $\frac{1}{5}$ or .2. Hence, 6 inches is $\frac{1}{5}$, or .2, of 30 inches.

As a practical example of this kind of comparison may be mentioned the case of two pulleys that are to be connected by a belt. It is here necessary to know the relation between the diameters of the two pulleys in order that their relative speeds may be calculated.

4. The two numbers compared must be units of the same denomination. For instance, if one number is given in inches, the other number must also be in inches; thus, 30 inches may be compared with 6 inches, as in the preceding example; but, 30 inches cannot be compared with 6 pounds. From the preceding remarks it will be seen that the term **ratio** means a comparison of two numbers of the same denomination or kind. The operation of comparing two numbers is called *finding the ratio* of the numbers.

5. Expressing a Ratio.—A ratio may be written in two different ways, both of which are correct. Thus, the ratio of 20 to 4, or the value of 20 compared to the value of 4, may be written 20:4 or $\frac{20}{4}$. Each of these expressions is read *the ratio of 20 to 4*. The ratio of 4 to 20 would be written either 4:20 or $\frac{4}{20}$. The method most commonly used in writing ratios is the first one shown; that is, the two numbers are separated by the sign (:), which is really an abbreviation of ÷, the sign of division, and has the same meaning, division being practically a method of indicating ratio. Hence, 20:4 is equal to 20÷4=5. In calculations, ratios are frequently written in the form of fractions; thus, the ratio of 20 to 4 may be written $\frac{20}{4}$.

6. Terms of Ratio.—The two numbers to be compared are known as the **terms** of the ratio; thus, in the ratio 30:6, 30 and 6 are the two terms. When both terms are considered together they are called a **couplet;** when considered separately, the first term is called the **antecedent,** and the second term the **consequent.** Thus, in the ratio 30 : 6, 30 and 6 form a couplet, in which 30 is the antecedent and 6 the consequent.

7. **Simple Ratio.**—When a ratio has only one antecedent and one consequent, it is known as a **simple ratio.** Thus, the ratio 30 : 6 is a simple ratio.

8. **Direct and Inverse Ratio.**—When it is desired to compare two denominate numbers of the same kind, the comparison is usually made by finding the ratio of the first number to the second. This is known as a **direct ratio.** For instance, the direct ratio of 22 feet to 9 feet is 22 : 9. If the given terms are interchanged the ratio becomes an **inverse ratio.** Thus, the inverse ratio of 22 feet to 9 feet is 9 : 22. The direct ratio of 5 pounds to 11 pounds is 5 : 11, and the inverse ratio is 11 : 5.

Every ratio is understood to be a *direct* ratio unless otherwise stated.

9. **Value of a Ratio.**—Distinction must be made between a *ratio* and its *value.* A ratio is represented by its two terms. The **value** of a ratio is the quotient obtained by dividing the first term by the second. Thus, the ratio of 20 to 4 is 20 : 4; the value of this ratio is $20 \div 4 = 5$.

10. **Ratio Considered as a Fraction.**—By expressing the ratio in the fractional form, for example, the ratio of 18 to 3 as $\frac{18}{3}$, it follows from the laws of fractions that both terms may be multiplied or both divided by the same number, without altering the value of the ratio. Thus,

$$\frac{18}{3} = \frac{18 \times 4}{3 \times 4} = \frac{72}{12}; \quad \frac{18}{3} = \frac{18 \div 3}{3 \div 3} = \frac{6}{1}$$

In each case the value of the ratio is 6.

In the ratio 6 : 10, in which the antecedent is the smaller one of the two terms, the ratio, when expressed in the fractional form, becomes $\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$. Hence, the value of the ratio

$$6 : 10 \text{ is } \frac{3}{5}.$$

When a ratio is given in the fractional form a direct ratio may be changed to an inverse ratio by simply inverting the fraction. For instance, the direct ratio of 21 to 7 is $\frac{21}{7}$, which has a value of 3. The inverse ratio of 21 to 7 is $\frac{7}{21}$; in this case the value of the ratio is $\frac{1}{3}$, which is the reciprocal of 3. An inverse ratio is, therefore, also called a **reciprocal ratio,** as its value is the reciprocal of the value of the direct ratio. The *reciprocal* of a number is 1 divided by that number; the reciprocal of a fraction is the fraction inverted.

11. Reducing a Ratio to Its Lowest Terms.—It is preferable that a ratio be reduced to its lowest terms and, if possible, that one of its terms be made unity, or 1. Thus, instead of saying that the ratio of the resistances of two electrical conductors is $\frac{5}{50}$, the ratio may be reduced to $\frac{1}{10}$; such an expression is often used instead of giving the actual values of the respective resistances.

A ratio is reduced to its lowest terms by dividing both terms by the same number until no number except 1 can be found that will divide both terms without a remainder. Thus, the ratio $\frac{84}{48}$, when reduced to its lowest terms, becomes $\frac{7}{4}$, which is obtained by dividing each number by 12. The operation may be written $\frac{84 \div 12}{48 \div 12} = \frac{7}{4}$.

12. Rules for Finding Direct and Inverse Ratio of Two Numbers.—The following rules may be used in cases where the direct or the inverse ratio of two numbers is to be found:

Rule I.—*The direct ratio of two numbers is found by making the first one of the given numbers the antecedent and the second one the consequent of the ratio.*

Rule II.—*The inverse ratio of two numbers is found by making the second number of those given the antecedent and the first one the consequent of the ratio.*

EXAMPLE 1.—(*a*) What is the direct ratio of 9 to 3? (*b*) What is the value of the ratio?

SOLUTION.—(*a*) Applying rule I, the first number, 9, is made the antecedent and the second number, 3, the consequent of the ratio; thus, the direct ratio of 9 to 3 is 9 : 3, or $\frac{9}{3}=\frac{3}{1}$. Ans.

(*b*) From Art. **9** the value of the ratio is $3 \div 1 = 3$. Ans.

EXAMPLE 2.—(*a*) What is the inverse ratio of 10 to 5? (*b*) What is the value of the ratio?

SOLUTION.—(*a*) Applying rule II, the second number, 5, is placed as the antecedent of the ratio, and 10 as the consequent; thus, the inverse ratio of 10 to 5 is 5 : 10, or $\frac{5}{10}=\frac{1}{2}$. Ans.

(*b*) From Art. **9** the value of the ratio is $1 \div 2 = .5$, or $\frac{1}{2}$. Ans.

EXAMPLE 3.—(*a*) Reduce the ratio 19 : 76 to its lowest terms. (*b*) What is the value of the ratio?

SOLUTION.—(*a*) According to Art. **11** a ratio is reduced to its lowest terms by dividing both terms by the same number, continuing the process until no number except 1 can be found that will be contained in each term without a remainder. In this case the terms of the ratio may be divided by 19; thus $\frac{19 \div 19}{76 \div 19}=\frac{1}{4}$. Ans.

(*b*) The value of the ratio is $\frac{1}{4}$, or .25. Ans.

EXAMPLE 4.—What is the inverse ratio of 8 and 72, and what is the value of the ratio?

SOLUTION.—Applying rule II, the inverse ratio of 8 to 72 is 72 : 8, which reduced to its lowest terms is 9 : 1. The value of the ratio is $9 \div 1 = 9$. Ans.

EXAMPLE 5.—What is the direct ratio of 72 to 8, and what is the value of the ratio?

SOLUTION.—Applying rule I, the direct ratio of 72 to 8 is 72 : 8 = 9 : 1. The value of the ratio is $9 \div 1 = 9$. Ans.

NOTE.—Examples 4 and 5 show that the direct and the inverse ratio of two numbers may be equal, depending on their relative positions in the statement contained in the example; for instance, whether 8 precedes or follows 72.

EXAMPLE 6.—A pair of gear-wheels contain 60 and 35 teeth, respectively. (*a*) What is the ratio of the number of teeth in the larger to the number of teeth in the smaller? (*b*) What is the ratio of the smaller number to the larger?

SOLUTION.—(*a*) The ratio of the larger number of teeth to the smaller is the ratio of 60 to 35, or 60 : 35, and reducing this to its lowest terms by dividing both terms by 5, the ratio becomes 12 : 7. Ans.

(*b*) The ratio of the smaller number to the larger is 35 : 60, which, reduced to its lowest terms by dividing both terms by 5, is equal to 7 : 12. Ans.

EXAMPLES FOR PRACTICE

1. What is the ratio of 126 to 18, reduced to its lowest terms? Ans. 7 : 1

2. Two gears have 39 and 54 teeth, respectively. What is the value of the ratio of the larger number to the smaller? Ans. $1\frac{5}{13}$

3. What is the value of the ratio of 6.25 to .75? Ans. $8\frac{1}{3}$

4. One pulley is 24 inches in diameter and another is 60 inches in diameter. What is the inverse ratio of the diameter of the smaller pulley to that of the larger? Ans. 5 : 2

5. A man traveled 250 miles, partly by rail and partly by boat. If he traveled 150 miles by rail, state (*a*) the distance traveled by boat; (*b*) the ratio of the distance traveled by rail to that traveled by boat.

Ans. { (*a*) 100 mi.
(*b*) 3 : 2

6. During one year A and B invest 5 dollars and 4 dollars, respectively, every month. (*a*) What is the ratio of A's investment to that of B? (*b*) What is the ratio of B's investment to that of A? (*c*) What are the values of the two ratios?

Ans. { (*a*) 5 : 4
(*b*) 4 : 5
(*c*) $1\frac{1}{4}$ and $\frac{4}{5}$

PROPORTION

SIMPLE PROPORTION

DIRECT AND INVERSE PROPORTIONS

13. Elements of a Proportion.—A **simple proportion** consists of two simple ratios of the same value connected by an equality sign (=) or a double colon (: :). For example, the ratios 8 : 6 and 12 : 9 are of the same value, and a proportion may be formed by them, as 8 : 6 = 12 : 9, or 8 : 6 : : 12 : 9. The equality sign is used more frequently than the double colon, so the proportions in this and other Sections will be written with the equality sign. The proportion 8 : 6 = 12 : 9 is read *8 is to 6 as 12 is to 9*, or *the ratio of 8 to 6 is equal to the ratio of 12 to 9*. This same proportion can also be written $\frac{8}{6}=\frac{12}{9}$, each of the two ratios being given as a fraction.

The term **couplet** used in connection with a ratio is sometimes applied to the elements of a proportion. Each ratio, or couplet, has two terms and the terms of a proportion are called *first*, *second*, *third*, and *fourth*, numbering from left to right. The first and fourth terms are the **extremes**; the second and third terms are the **means**.

The following table gives the proportion 25 : 10 = 40 : 16 and the terms applied to its various elements, or parts.

Ratios, or couplets	First				Second		
Terms..............	First		Second		Third		Fourth
Quantities..........	25	:	10	=	40	:	16
	Extreme		*Mean*		*Mean*		*Extreme*

It is seen that 25 : 10 is the first couplet or ratio, and that 40 : 16 is the second couplet or ratio. Numbering from left to right, 25 is the first and 16 the fourth term. It is also seen that the numbers 25 and 16 form the two extremes and the numbers 10 and 40 the two means of the proportion.

14. Use of Proportions.—In practice there are numerous cases in which one ratio is given and it is required to find an equal ratio of which one term is already known; this is the purpose of a proportion, as may be made clearer by an example. Let it be supposed that the top of a table is 6 feet long and 3 feet wide and that it is required to make another table the top of which is 8 feet long, the ratio of length to width being the same as in the smaller table. Here the first ratio is $6:3$ and the second ratio $8:x$, the missing term being indicated by the symbol x. The value of the first ratio is $6\div3=2$ and if the value of the second ratio is to be 2, also, it is evident that the missing term, x, must be $8\div2=4$, and the new table top must, therefore, be 4 feet wide. Hence, the complete proportion is $6:3=8:4$.

15. A problem frequently met with in practice may be given as another example. The ratio of the weights of the ingredients in a certain mixture is $7:3$. Of the first ingredient there is on hand 91 pounds, and it is required to know what weight of the second ingredient will be required in order that the ratio of the two weights may be as $7:3$. The first ratio is $7:3$ and the second is $91:x$. In this case the relation between the two ratios is so simple that the problem may be solved by inspection in the following manner: It is seen that 7 is contained 13 times in 91; according to Art. **10** both terms of a ratio may be multiplied by the same number without altering the value of the ratio. Hence, on multiplying the terms of the ratio $7:3$ by 13, the result is $7\times13:3\times13=91:39$. The numbers 91 and 39 are, therefore, the terms of the second ratio, showing that x, or the weight of the second ingredient, is equal to 39 pounds.

These examples will suffice to show some of the simpler cases in which the principle of proportion is applied in practice. Further explanations will show that the subject of proportion forms one of the most useful sections of arithmetic.

16. Direct and Inverse Proportions.—A **direct proportion** is one in which both couplets are direct ratios. A proportion is always understood to be direct unless the statement of a problem clearly indicates otherwise. An **inverse proportion** is one that requires one of the couplets to be expressed as an inverse ratio. Thus, if 8 is to 4 inversely as 3 is to x, one of the ratios must be reversed (it does not matter which one) and the proportion may be written in either of the following ways: $8:4=x:3$, or $4:8=3:x$. In the first proportion the second couplet is reversed; in the second proportion the first couplet is reversed.

17. Directly Proportional and Inversely Proportional Quantities.—In technical literature, one quantity is said to be *proportional* to another, or to *vary with* it, or to *increase with* it, or to *decrease with* it; any one of these expressions means that a change in one of the quantities causes, or is accompanied by, a corresponding change in the other. If the word *inversely* is used with one of these expressions, the meaning is that a change in one quantity causes an opposite change in the other. Sometimes the word *directly* is used when it is desirable to make sure that the meaning will not be understood to be inverse. For example, the resistance of an electric wire is directly proportional to its length, or varies with, or increases with, its length; that is, the longer the wire, the greater is its resistance. But, if a number of men are engaged in building a fence, the time required to finish the fence is *inversely* proportional to the number of men working on it; that is, the more men, the shorter the time.

18. Rules for Finding Unknown Terms of Proportions.—In any proportion, *the product of the extremes is equal to the product of the means.* For example, in the proportion $17:51=14:42$, the extremes are 17 and 42 and the means are 51 and 14. According to the preceding statement the product 17×42 must be equal to 51×14, which is true because $17\times 42=714$ and $51\times 14=714$. This important principle makes it possible to find an unknown term of a proportion when the three other terms are known. The following rules are based on this principle:

Rule I.—*To find an unknown extreme, divide the product of the means by the given extreme.*

Rule II.—*To find an unknown mean, divide the product of the extremes by the given mean.*

19. Direct Proportion.—In forming the two ratios of a proportion it is important to see that both terms of each ratio are of the *same* kind. For instance, if one term is given in pounds, the other must also be in pounds. But the four terms of a proportion need not be of the same kind; in one ratio both terms may be in pounds, and in the other both in feet, and so forth.

Another point to be noted in forming a direct proportion is that the terms are arranged so that the first term of each ratio refers to *one* of the things compared and that the second term in each ratio refers to the *other* thing compared. For instance, an example states that the weights of two parcels of sugar are 2 and 5 pounds, respectively, and that the cost of the smaller parcel is 12 cents. It is required to find the cost of the larger parcel. Here, one ratio is 2 : 5, and the other must be 12 : x, as the term 2 in the first ratio refers to the parcel that costs 12 cents; hence, 12 must be the first term of the second ratio, and the proportion is written

$$2 : 5 = 12 : x$$

20. Arrangement and Solution of Direct Proportions.—In the following method for solving proportions, it is found convenient always to let the unknown term, x, occupy the position of the *fourth* term in the proportion, and for the *third* term to write the number that is of the same kind as the required fourth term.

The conditions given in the example are now examined to ascertain whether the fourth term, x, will be larger or smaller than the third one. If *larger*, the larger number of those that are to form the first ratio is written as its *second* term, and the remaining number is written as the *first* term.

The preceding method of procedure is embodied in the following rule:

Rule.—*Make the unknown term, x, the fourth term of the proportion, and for the third term write the number that is of a* *similar kind. If the fourth term will be larger than the third, the second term must be larger than the first; or if the fourth term will be smaller than the third, the second term must be smaller than the first.*

EXAMPLE 1.—If 7 pounds of putty costs 56 cents, what will be the cost of 36 pounds?

SOLUTION.—In this example the cost of 36 pounds of putty is the required term, and is, therefore, written as the fourth term, x. The other number of the same kind is 56 cents. So, the second ratio of the proportion must be written $56 : x$.

As 36 pounds of putty must cost more than 7 pounds, it follows that the term x will be larger than the third term, 56. Hence, according to the rule, the second term must be larger than the first. The proportion is, therefore, written

$$7 : 36 = 56 : x$$

In this case one of the extremes is not known. Hence, by rule I, Art. **18**, the unknown extreme, x, is equal to the product of the means divided by the known extreme, or

$$x=\frac{36\times 56}{7}=288 \text{ cents, or } \$2.88. \quad \text{Ans.}$$

EXAMPLE 2.—The weights of two patterns, made of pine, are in the ratio of 2 to 3, and the iron casting made from the smaller pattern weighs 42 pounds. What is the weight of the iron casting made from the larger pattern?

SOLUTION.—The weight of the casting made from the larger pattern is unknown and is therefore indicated by x. The other number of the same kind is 42 pounds. So, the second couplet of the proportion must be written $42 : x$.

The example states that the weights of the patterns are as $2 : 3$, and the weight of the casting made from the lighter pattern is 42 pounds; it follows that the casting made from the other pattern must be heavier. According to the rule, the second term must be larger than the first and the proportion is written

$$2 : 3 = 42 : x$$

By rule I, Art. **18**, the unknown extreme, x, is equal to the product of the means divided by the known extreme, or

$$x=\frac{3\times 42}{2}=\frac{126}{2}=63$$

That is, the casting made from the larger pattern weighs 63 lb. Ans.

EXAMPLE 3.—The ratio of the numbers of teeth in two gear-wheels is as 2 to 5. If the number of teeth in the second or larger wheel is 90, how many teeth are there in the smaller wheel?

SOLUTION.—The unknown term, x, is here the number of teeth on the smaller wheel and is put down as the fourth term in the proportion. The other number of the same kind is 90 teeth, which is made the third term in the proportion. So the second ratio is written

$$90 : x$$

It is seen from the example that the term 2 refers to the unknown number of teeth. As 2 is smaller than 5, it follows that x must be smaller than 90. Hence, the second term must be smaller than the first and the proportion is written

$$5 : 2 = 90 : x$$

By rule I, Art. **18**,

$$x = \frac{2 \times 90}{5} = \frac{180}{5} = 36 \text{ teeth. Ans.}$$

NOTE.—In all problems involving fractions cancelation should be resorted to wherever possible so as to simplify operations. If in these and other examples cancelation is omitted it is for the purpose of concentrating attention on the one process under consideration.

21. Arrangement and Solution of Inverse Proportions.—Many problems in proportion are so stated that the first and third terms in the complete proportion refer to the same thing. This is the case with the preceding examples. For instance, in the proportion given as a solution of example 1, Art. **20**, the first and the third terms, 7 and 56, stand for 7 pounds of putty at a cost of 56 cents. The second and the fourth terms refer to the cost of 36 pounds of putty. Such proportions are known as *direct* proportions.

But the conditions stated in the problems are not always such as to give proportions of this kind. Sometimes, it is found, on examination of the complete proportion, that the first and the third terms do *not* refer to the same thing. Such proportions are known as *inverse* proportions. It must, however, be clearly understood that whether a proportion is direct or inverse does not in any way interfere with the application of the rule in Art. **20**, as will be seen from the following examples in which the solution is in every case found by means of inverse proportion.

EXAMPLE 1.—If 3 men can do a certain job in 20 days, how long will it take 12 men to do a similar job, working at the same rate?

SOLUTION.—The unknown term is the number of days required to do a certain job; it is written as the fourth term. The number of a similar kind is 20 days, which is written as the third term. The second ratio of the proportion is, therefore, $20 : x$.