ALGEBRA

(PART 4)

Serial 379D ——— Edition 1

FRACTIONS

1. A **fraction** is an indicated division in which the dividend is written over the divisor with a line between them.

Thus, $\frac{a}{x}$, $\frac{a+y}{z}$, $\frac{c}{a+b}$, $\frac{x+y}{x-y}$ are fractions.

The dividend is the **numerator** and the divisor the **denominator** of the fraction. The numerator and the denominator are the **terms** of the fraction.

2. In arithmetic, a proper fraction is a fraction whose numerator is less than its denominator. In algebra, a **proper fraction** is a fraction whose numerator is of lower degree than its denominator.

Thus, $\frac{5x+6}{2x^2+3}$ and $\frac{x-1}{x^2+1}$ are proper fractions.

3. In arithmetic, an improper fraction is a fraction whose numerator is not less than its denominator. In algebra, an **improper fraction** is a fraction whose numerator is not of lower degree than its denominator.

Thus, $\frac{3x+14}{11x+19}$, $\frac{x^2+5x+9}{3x-7}$ are improper fractions.

4. A **mixed number** or a **mixed expression** consists of an integral and a fractional part.

Thus, $x^2+3x+1+\frac{x-1}{x^2+1}$ is a mixed expression.

5. Any integral number may be written in the *fractional form* with a denominator 1.

Thus, $7 = \frac{7}{1}$; $x+1 = \frac{x+1}{1}$; $a+b = \frac{a+b}{1}$; etc.

6. *The value of a fraction is not changed if its terms are both multiplied or both divided by the same number, not 0.*

The principle may be illustrated as follows: Let AE, Fig. 1, be a line 1 inch in length that is divided into four equal parts at the points B, C, and D. Then AD contains three of these equal parts; that is, $AD = \frac{3}{4}$ inch. Let each of the equal parts AB, BC, CD, and DE be divided into two equal parts by the points P, Q, R, and S. Then the whole line AE is divided into eight equal parts and AD contains six of these equal parts.

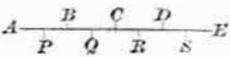

FIG. 1

Therefore, $AD = \frac{6}{8}$ inch

But, $AD = \frac{3}{4}$ inch

Hence, $\frac{3}{4} = \frac{6}{8}$

or $$\frac{3}{4} = \frac{3\times 2}{4\times 2}$$

That is, the value of the fraction $\frac{3}{4}$ is not changed by multiplying both of its terms by 2; nor is the value of the fraction $\frac{6}{8}$ changed by dividing both of its terms by 2. This principle may be proved as follows:

Let $$p = \frac{a}{b} \qquad (1)$$

Multiplying both terms of equation (1) by b,

$$bp = \frac{a}{b}\times b$$

or $$bp = a \qquad (2)$$

Multiplying both terms of equation (2) by n,

$$bnp = an \qquad (3)$$

Now divide both terms of equation (3) by bn,

$$bnp \div bn = an \div bn$$

or $$p = \frac{an}{bn}$$

But, $$p = \frac{a}{b}$$

Therefore, $$\frac{a}{b} = \frac{an}{bn}$$

That both terms of a fraction can be divided by the same number (not 0) can be proved in a similar manner.

7. Sign of a Fraction.—The sign of a fraction is written to the left of the line that separates the numerator from the denominator. When no sign is written, + is to be understood. This sign is distinct from the sign of both numerator and denominator.

Thus, $+\frac{+a}{+b}$; $+\frac{-a}{+b}$; $-\frac{a-b}{a+b}$

8. The dividing line of a fraction has the effect of a parenthesis on both the numerator and the denominator.

Thus, $$\frac{a-b}{a+b} = \frac{(a-b)}{(a+b)}$$

It is a good plan, when operating with a fraction, to enclose both the numerator and the denominator in parentheses, and when the operations are completed, to remove them.

9. Law of Signs.—From the law of signs for multiplication and division, it follows that:

1. *Changing the sign of both the numerator and the denominator of a fraction does not change the sign of the fraction.*

Thus, $\frac{+x}{+y} = \frac{-x}{-y}$; and $\frac{-x}{+y} = \frac{+x}{-y}$

This can be shown to be true by the principle of Art. **6.**

Thus, $\frac{+x\times-1}{+y\times-1} = \frac{-x}{-y}$; and $\frac{-x\times-1}{+y\times-1} = \frac{+x}{-y}$

2. *Changing the sign of either the numerator or the denominator of a fraction changes the sign of the fraction.*

Thus, $$\frac{+x}{+y} = -\frac{-x}{+y};\ \frac{-x}{+y} = -\frac{+x}{+y}$$

$$\frac{a}{a-x} = -\frac{a}{-(a-x)} = -\frac{a}{x-a}$$

10. By the law of signs for multiplication, to change an even number of signs of the factors of a product does not change the sign of the product.

Thus, $a \times b \times c = (-a) \times b \times (-c)$

Putting this statement with those of Art. **9** gives the principles:

The signs of an **even** *number of factors of the numerator and denominator of a fraction may be changed without changing the sign of the fraction.*

The signs of an **odd** *number of factors of the numerator and denominator of a fraction may be changed if the sign of the fraction is changed.*

EXAMPLE 1.—Change the fraction $-\frac{x-y+z}{r}$ into an equivalent fraction having a positive sign.

SOLUTION.—If the sign of the numerator of the fraction is changed the sign of the fraction must be changed.

Therefore,

$$-\frac{x-y+z}{r} = +\frac{-(x-y+z)}{r} = \frac{-x+y-z}{r} = \frac{y-z-x}{r} \quad \text{Ans.}$$

EXAMPLE 2.—Change the fractions $-\frac{x}{x^2-1}$ and $\frac{1}{1-x^2}$ into fractions having like denominators.

SOLUTION.—By the principle of this article,

$$-\frac{x}{x^2-1} = +\frac{x}{-(x^2-1)} = \frac{x}{-x^2+1} = \frac{x}{1-x^2}$$

Thus, $-\frac{x}{x^2-1}$ is changed into the equivalent fraction $\frac{x}{1-x^2}$ whose denominator is like the denominator of the fraction $\frac{1}{1-x^2}$.

EXAMPLE 3.—Change the fractions

$$\frac{1}{(a-b)(b-c)(c-a)}, \quad -\frac{a}{(a-b)(c-b)(c-a)}, \text{ and } \frac{b}{(a-b)(c-b)(a-c)}$$

into equivalent fractions having like denominators.

SOLUTION.—By changing the $c-b$ in the denominator of the second fraction into $b-c$, the denominators of the first two fractions will be like.

Thus,

$$-\frac{a}{(a-b)(c-b)(c-a)} = \frac{a}{(a-b)[-(c-b)](c-a)} = \frac{a}{(a-b)(b-c)(c-a)}$$

By changing two factors of the denominator of the third fraction,

$$\frac{b}{(a-b)(c-b)(a-c)} = \frac{b}{(a-b)[-(c-b)][-(a-c)]} = \frac{b}{(a-b)(b-c)(c-a)}$$

Hence, the three given fractions are equivalent to

$$\frac{1}{(a-b)(b-c)(c-a)}, \quad \frac{a}{(a-b)(b-c)(c-a)}, \text{ and } \frac{b}{(a-b)(b-c)(c-a)} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Change each of the following fractions into equivalent fractions with opposite signs, leaving the denominators unchanged.

1. $-\frac{a+b-c}{x+y+z}$ Ans. $\frac{c-a-b}{x+y+z}$

2. $\frac{(x-2)(1-x)}{(x+2)(x+1)}$ Ans. $-\frac{(x-2)(x-1)}{(x+2)(x+1)}$

3. $\frac{(a-b)(b+c)}{(x+a)(x+b)}$ Ans. $-\frac{(b-a)(b+c)}{(x+a)(x+b)}$

Change each of the following groups of fractions into equivalent fractions having like denominators.

4. $\frac{a}{x+y-z}, \frac{b}{z-x-y}$ Ans. $\frac{a}{x+y-z}, -\frac{b}{x+y-z}$

5. $\frac{1}{(x-a)(b-x)(x-c)}, \frac{1}{(a-x)(x-b)(c-x)}, \frac{1}{(x-a)(x-b)(c-x)}$

Ans. $-\frac{1}{(x-a)(x-b)(x-c)}, \frac{1}{(x-a)(x-b)(x-c)}, -\frac{1}{(x-a)(x-b)(x-c)}$

REDUCTION OF FRACTIONS

11. To Reduce an Integral Expression to an Improper Fraction.

EXAMPLE.—Reduce $x+a$ to an improper fraction having $x-a$ for its denominator.

SOLUTION.—By Art. **5**, $x+a=\frac{x+a}{1}$

Then, by Art. **6**,

$$\frac{x+a}{1}=\frac{(x+a)(x-a)}{1(x-a)}=\frac{x^2-a^2}{x-a} \text{ Ans.}$$

12. Rule.—*To reduce an integral expression to a fraction having a given denominator, multiply the integral expression by the given denominator and write the product over the denominator.*

EXAMPLE.—Reduce a^2-ab+b^2 to an improper fraction having $a+b$ for its denominator.

SOLUTION.—Multiplying a^2-ab+b^2 by $a+b$ gives a^3+b^3.

Hence. $a^2-ab+b^2=\frac{a^2-ab+b^2}{1}\times\frac{a+b}{a+b}=\frac{a^3+b^3}{a+b}$ Ans.

13. To Reduce a Mixed Expression to an Improper Fraction.

EXAMPLE.—Reduce $x - a + \frac{a^3}{x^2 + ax + a^2}$ to an improper fraction.

SOLUTION.—By Art. 5, write the expression in the form

$$\frac{x-a}{1} + \frac{a^3}{x^2 + ax + a^2}$$

By Art. 12,

$$\frac{x-a}{1} + \frac{a^3}{x^2 + ax + a^2} = \frac{(x-a)(x^2 + ax + a^2)}{x^2 + ax + a^2} + \frac{a^3}{x^2 + ax + a^2}$$

$$= \frac{(x-a)(x^2 + ax + a^2) + a^3}{x^2 + ax + a^2} = \frac{x^3 - a^3 + a^3}{x^2 + ax + a^2}$$

$$= \frac{x^3}{x^2 + ax + a^2} \text{ Ans.}$$

14. Rule.—*To reduce a mixed expression to a fraction: By changing the sign of the numerator, make the sign of the fraction + if it is −; multiply the integral part by the denominator of the fraction, add the product to the numerator, and write the result over the denominator.*

EXAMPLE 1.—Reduce $m^2 + mn + n^2 - \frac{1}{m-n}$ to a fraction.

SOLUTION.—By Art. 9,

$$m^2 + mn + n^2 - \frac{1}{m-n} = m^2 + mn + n^2 + \frac{-1}{m-n}$$

Multiplying $m^2 + mn + n^2$ by $m - n$ gives $m^3 - n^3$. Then adding -1 to this result, gives $m^3 - n^3 - 1$.

Hence, $$m^2 + mn + n^2 - \frac{1}{m-n} = \frac{m^3 - n^3 - 1}{m-n} \text{ Ans.}$$

EXAMPLE 2.—Reduce $a + 2 - \frac{8 - a^3}{a^2 - 2a + 4}$ to a fraction.

SOLUTION.—By Art. 9,

$$a + 2 - \frac{8 - a^3}{a^2 - 2a + 4} = a + 2 + \frac{a^3 - 8}{a^2 - 2a + 4}$$

Multiplying $a + 2$ by $a^2 - 2a + 4$ gives $a^3 + 8$. Adding $a^3 + 8$ and $a^3 - 8$, gives $2a^3$.

Hence, $$a + 2 - \frac{8 - a^3}{a^2 - 2a + 4} = \frac{2a^3}{a^2 - 2a + 4} \text{ Ans.}$$

EXAMPLES FOR PRACTICE

Reduce to improper fractions:

1. $x + \frac{2}{x+3}$ Ans. $\frac{x^2 + 3x + 2}{x+3}$

2. $a+b-\dfrac{a^2+b^2}{a-b}$ Ans. $\dfrac{2b^2}{b-a}$

3. $x^3+x^2+x+1+\dfrac{2}{x-1}$ Ans. $\dfrac{x^4+1}{x-1}$

4. $x^3-x^2+x-1+\dfrac{1}{x+1}$ Ans. $\dfrac{x^4}{x+1}$

5. $x^2-2xy+4y^2-\dfrac{6y^3}{x+2y}$ Ans. $\dfrac{x^3+2y^3}{x+2y}$

6. $1-\left(a-a^2+\dfrac{1}{a+1}\right)$ Ans. $\dfrac{a^3}{a+1}$

15. To Reduce an Improper Fraction to an Integral or a Mixed Expression.

Example 1.—Reduce $\dfrac{23a}{7}$ to an integral or mixed expression.

Solution.—Performing the division indicated gives $3a$ as quotient and $2a$ as remainder. Complete the division by writing $2a$ over 7, and then add the fraction thus found to the quotient $3a$.

$$\frac{23a}{7} = 23a \div 7 = 3a + \frac{2a}{7} \quad \text{Ans.}$$

Example 2.—Reduce $\dfrac{x^3-2xy+y^3}{x+y}$ to an integral or mixed expression.

Solution.—Performing the division indicated by the fraction gives x^2-xy+y^2 as quotient and $-2xy$ as remainder. Indicate that $-2xy$ is to be divided by $x+y$ by writing $-2xy$ over $x+y$, and then add the fraction thus formed to the integral part of the quotient.

$$\begin{array}{l|l}
x^3-2xy+y^3 & x+y \\
\underline{x^3+x^2y} & \overline{x^2-xy+y^2} \\
\quad -x^2y-2xy & \\
\quad \underline{-x^2y-xy^2} & \\
\qquad xy^2-2xy+y^3 & \\
\qquad \underline{xy^2 \qquad\quad +y^3} & \\
\qquad\quad -2xy &
\end{array}$$

Therefore, $\dfrac{x^3-2xy+y^3}{x+y}=x^2-xy+y^2+\dfrac{-2xy}{x+y}$ Ans.

Note.— $x^2-xy+y^2+\dfrac{-2xy}{x+y}$ may be written in the form $x^2-xy+y^2-\dfrac{2xy}{x+y}$. See Art. 9.

16. Rule.—*To reduce an improper fraction to an integral or a mixed expression: Divide the numerator by the denominator; if there is a remainder, write it over the denominator and add the fraction thus formed to the integral part of the quotient.*

EXAMPLE.—Reduce $\frac{x^2+5x+6}{x+4}$ to a mixed expression.

SOLUTION.—Dividing the numerator by the denominator gives $x+1$ as quotient and 2 as remainder. Writing the 2 over the denominator $x+4$ and adding the fraction thus formed to the integral part of the quotient gives $x+1+\frac{2}{x+4}$.

$$\begin{array}{l} x^2+5x+6\,(x+4 \\ \underline{x^2+4x}\quad (x+1 \\ \qquad x+6 \\ \qquad \underline{x+4} \\ \qquad\quad 2 \end{array}$$

$$\therefore\ \frac{x^2+5x+6}{x+4}=x+1+\frac{2}{x+4}\quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Reduce to integral or mixed expressions:

1. $\frac{27xy+4y}{9}$ Ans. $3xy+\frac{4y}{9}$
2. $\frac{8x^2+3y}{4x}$ Ans. $2x+\frac{3y}{4x}$
3. $\frac{a^2+3a+2}{a+3}$ Ans. $a+\frac{2}{a+3}$
4. $\frac{x^3-2x^2}{x^2-x+1}$ Ans. $x-1-\frac{2x-1}{x^2-x+1}$
5. $\frac{x^3+ax^2-3a^2x-3a^3}{x-2a}$ Ans. $x^2+3ax+3a^2+\frac{3a^3}{x-2a}$
6. $\frac{5x^3-x^2+5}{5x^2+4x-1}$ Ans $x-1+\frac{5x+4}{5x^2+4x-1}$

17. To Reduce a Fraction to Its Lowest Terms.

A fraction is in its **lowest terms** when its numerator and denominator have no common factor.

The reduction of a fraction to its lowest terms depends on Art. **6.**

EXAMPLE 1.—Reduce $\frac{9a^3bc^2}{12a^2b^2c^3}$ to its lowest terms.

SOLUTION.—Separating each term of the fraction into two factors one of which is their highest common divisor, and omitting this common factor, gives the fraction in its lowest terms.

$$\frac{9a^3bc^2}{12a^2b^2c^3}=\frac{3a^2bc^2\times 3a}{3a^2bc^2\times 4bc}=\frac{3a}{4bc}\quad \text{Ans.}$$

NOTE.—Notice that omitting the highest common divisor from both terms of a fraction is equivalent to dividing both terms by their highest