ALGEBRA

(PART 3)

Serial 379C

Edition 1

FACTORS AND MULTIPLES

PRIME AND COMPOSITE NUMBERS

1. In multiplication, two or more factors are given to find their product; in division, the product and one factor are given to find the other factor. It is frequently necessary, however, to find the factors of a given product without knowing any of them.

2. In multiplication and division, any two numbers are regarded as factors of their product. Thus, 2.5 and 2 are factors of their product 5. But in treating factors and multiples, only integral numbers are considered and 2.5 and 2 are not regarded as factors of 5. The word *number*, when used in connection with factors and multiples, always means an *integral number*.

3. A **factor, measure,** or **submultiple** of a number is any exact divisor of that number.

Thus, 2, 3, 4, 6, 8, and 12 are all factors of 24.

A **multiple** of a number is any product of the number by an integer.

Thus, 12, 18, 24, etc. are multiples of 6.

4. Every number has two exact divisors, viz., the number itself and unity. For, if x denotes any number, $x \div x = 1$, and $x \div 1 = x$. But neither of these two divisors of a number is regarded as a factor.

5. A **prime number,** or simply a **prime,** is a number that has no factors except itself and unity.

Thus, 2, 3, 5, 7, 11, 13, 17, etc. are primes.

6. A prime number that is a factor of another number is a **prime factor** of that number.

Thus, 2 and 3 are prime factors of 6; 7 is a prime factor of 14, 21, etc.

7. Every number not a prime can be resolved into prime factors, and this resolution can give only one set of prime factors.

Thus, $3 \times 7 \times 13$ is the only set of prime factors of 273; also, $2 \times 5 \times 7 \times 11$ are the only prime factors of 770.

8. A **composite number** is a number that can be separated into two or more factors besides itself and unity.

WWW.LIBRUM.US

Thus, 8, 12, 20, 36, etc. are composite numbers.

9. The multiples of 2 are **even** numbers; all other numbers are **odd.**

Thus, 2, 4, 6, 8, etc. are even numbers; while 1, 3, 5, 7, etc. are odd numbers.

10. The prime factors of a given number must be found by trial. Beginning with 2 and taking each of the prime numbers in succession, we determine which of them are exact divisors of the given number.

EXAMPLE 1.—Find the prime factors of 534.

SOLUTION.—Dividing by 2 and then by 3,

$$534 = 2 \times 3 \times 89$$

$$\begin{array}{r} 2\underline{)534} \\ 3\underline{)267} \\ 89 \end{array}$$

As 2, 3, and 89 are primes, they are the prime factors of 534.

Thus, $534 = 2 \times 3 \times 89$. Ans.

EXAMPLE 2.—Find the prime factors of 862.

SOLUTION.—After dividing 862 by 2, try to divide 431 by 2, 3, 5, 7, 11, 13, 17, and 19. It is unnecessary to try any prime number greater than 19, for 431 divided by 19 gives a quotient less than 23, the next prime number. Therefore, if 431 were divisible by 23 or any number greater than 23, the quotient would be less than 19, and 431 would have a factor less than 19. But by trial it is found that 431 has no factor less than 19, and is, therefore, a prime number. Thus,

$$2\overline{)862}$$
$$431$$

$$862 = 2 \times 431. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Find the prime factors of:

1. 35 — Ans. 5×7
2. 117 — Ans. $3^2 \times 13$
3. 3,575 — Ans. $5^2 \times 11 \times 13$
4. 13,260 — Ans. $2^2 \times 3 \times 5 \times 13 \times 17$

FACTORS OF ALGEBRAIC EXPRESSIONS

11. An algebraic term is **integral** if it does not contain a *letter* as a *divisor;* otherwise it is **fractional.**

Thus, ab, x^2, $3mn^2x$, $\frac{2}{3}xy$ are integral terms; while $a \div b$, $\frac{x}{y}$, $\frac{3m}{4n}$ are fractional. An integral term may have either an integral or a fractional value; so also may a fractional term.

WWW.LIBRUM.US

The classification of terms into integral and fractional has reference to their literal part, not to their numerical part or to their numerical value.

An **integral expression** is an expression of which all the terms are integral.

Thus, $5x + 3x^2 + 6x^3 + 3acx^4$ is an integral expression. But $\frac{x}{y} + \frac{3a}{b} + \frac{4m}{x}$ is a fractional expression.

An expression is said to be *integral with respect to a certain letter* when that letter does not occur as a divisor in any term.

Thus, $\frac{x}{a} + \frac{x^2y}{a^2} + \frac{x^3y^2}{a^3}$ is integral with respect to x, but fractional with respect to a. In the following discussion of factors and multiples, only integral expressions are treated.

12. Factors of Monomials.—Since monomials containing more than one element are simply indicated multiplications, the factors of a monomial are found by mere inspection. Thus,

$$11a^4x^2 = 11 \times a \times a \times a \times a \times x \times x$$

13. Factors of Polynomials.—The product of two or more binomials or trinomials often assumes a certain type form, and when these type forms appear, it is easy to find the factors. Some of the simplest methods of finding these factors are given in the following articles.

CASE I

14. To Factor a Polynomial When All of Its Terms Have a Common Factor.—The common factor is found by inspection and the other factor is found by dividing the polynomial by the common factor.

EXAMPLE.—Find the factors of $16x^2y^2 + 4x^3y^2 - 12xy^4$.

SOLUTION.—It is evident that each term contains the common factor $4xy^2$. Dividing the number by $4xy^2$, the quotient is $4x + x^2 - 3y^2$, which is the other factor.

WWW.LIBRUM.US

Hence, $16x^2y^2 + 4x^3y^2 - 12xy^4 = 4xy^2(4x + x^2 - 3y^2)$. Ans.

15. To discover the monomial factor of a polynomial, first ascertain the factors common to all the numerical coefficients. Then examine the polynomial to find the letters common to every term, take each of these common letters with the lowest exponent it has in any term of the polynomial. The product of the letters so chosen and the common factors of the numerical coefficients is the monomial factor.

EXAMPLE.—Find the factors of $12ab^3c^3 - 18a^3c^2y + 24a^2c^4 - 36a^4bc^4y^2$.

SOLUTION.—The numerical coefficients are 12, 18, 24, and 36, and 2 and 3 are their only common prime factors. The letters a and c are common to all the terms, and the lowest power of a is the first, and of c the square. Therefore, the monomial factor is $2 \times 3 \times a \times c^2$, or $6ac^2$. Dividing the polynomial by $6ac^2$, the quotient is $2b^3c - 3a^2y + 4ac^2 - 6a^3bc^2y^2$. Hence,

$$12ab^3c^3 - 18a^3c^2y + 24a^2c^4 - 36a^4bc^4y^2 = 6ac^2(2b^3c - 3a^2y + 4ac^2 - 6a^3bc^2y^2).$$

Ans.

EXAMPLES FOR PRACTICE

Find the factors of:

1. $a^3 + ax$ Ans. $a(a^2 + x)$
2. $12a^4 - 2a^2 + 4a^3$ Ans. $2a^2(6a^2 - 1 + 2a)$
3. $30m^4n^2 - 6n^3$ Ans. $6n^2(5m^4 - n)$
4. $49a^2b^3c^4 - 63a^3b^2c^4 + 7a^4b^2c^3$ Ans. $7a^2b^2c^3(7bc - 9ac + a^2)$

CASE II

16. To Factor a Trinomial That Is a Perfect Square. A number is a **perfect square** when it is the product of two equal factors. Either of these equal factors is the **square root** of the number.

17. *Every square number is positive.* This is evident, since from the law of signs for multiplication like signs give plus. That is, any number, positive or negative, when multiplied by itself gives a positive result.

18. By multiplication,

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a-b)^2 = a^2 - 2ab + b^2$$

Here a may represent any number and b any other number. The term $2ab$ is equal to $2\sqrt{a^2} \times \sqrt{b^2}$. Hence,

A trinomial is a perfect square if two terms are perfect squares and the other term is equal to twice the product of the square roots of the square terms.

EXAMPLE 1.—Determine whether $9a^4b^6 + 4 - 6a^2b^3$ is a perfect square.

SOLUTION.—In this polynomial $9a^4b^6$ and $+4$ are perfect squares, but the other term is not equal to twice the product of the square roots of these two terms; that is, $2 \times 3a^2b^3 \times 2$ does not equal $6a^2b^3$, so that the given trinomial is not a perfect square.

EXAMPLE 2.—Determine whether $4a^4 + 4a^2b^2 - b^4$ is a perfect square.

SOLUTION.—Now $-b^4$ is not a perfect square because every perfect square has a + sign. But $4a^4$ and $+4a^2b^2$ are perfect squares. The square roots of these terms are $2a^2$ and $2ab$, but b^4 is not equal to twice the product of $2a^2$ and $2ab$; therefore, the given polynomial is not a perfect square.

19. Rule.—*To find the factors of a trinomial that is a perfect square: Extract the square roots of the square terms of the trinomial, and connect these square roots by the sign of the other term. This result is one of the factors and the polynomial is equal to two such factors.*

EXAMPLE 1.—Factor $36m^2 - 24mn + 4n^2$.

SOLUTION.—The square terms are $36m^2$ and $4n^2$, and their square roots are $6m$ and $2n$. The sign of the other term is $-$. Therefore, each factor is $6m - 2n$; whence,

$$36m^2 - 24mn + 4n^2 = (6m - 2n)^2. \quad \text{Ans.}$$

EXAMPLE 2.—Factor $9a^4b^2c^2 + 6a^2bc + 1$.

SOLUTION.—The square terms are $+9a^4b^2c^2$ and $+1$, and their square roots are $3a^2bc$ and 1. The sign of the other term is $+$.

Therefore, $9a^4b^2c^2 + 6a^2bc + 1 = (3a^2bc + 1)^2$. Ans.

EXAMPLES FOR PRACTICE

Determine which of the following are perfect squares and factor them:

1. $x^2 + xy + y^2$ — Ans. Not a square
2. $a^4 - 2a^2x^2 + x^4$ — Ans $(a^2 - x^2)^2$
3. $m^2 + 2mn - n^2$ — Ans. Not a square
4. $x^2 - 16x + 64$ — Ans. $(x - 8)^2$
5. $n^4 - 26n^2 + 169$ — Ans. $(n^2 - 13)^2$
6. $25x^2 + 70xyz + 49y^2z^2$ — Ans. $(5x + 7yz)^2$
7. $2mx + m^2 + x^2$ — Ans. $(m + x)^2$
8. $1 - 2ab^2c^3 + a^2b^4c^6$ — Ans. $(1 - ab^2c^3)^2$

WWW.LIBRUM.US

CASE III

20. To Factor the Difference Between Two Perfect Squares.

By multiplication,

$$(a + b)(a - b) = a^2 - b^2$$

As a and b may represent any numbers, hence the following rule.

21. Rule.—*To find the factors of a number that is the difference between two squares, extract the square root of each term, add the roots for the first factor and subtract the second from the first for the second factor.*

EXAMPLE 1.—Factor $9x^8y^6 - 4$.

SOLUTION.—The square roots of the terms are $3x^4y^3$ and 2. The sum of these roots is $3x^4y^3 + 2$ and the remainder, when the second is subtracted from the first, is $3x^4y^3 - 2$.

Hence, $9x^8y^6 - 4 = (3x^4y^3 + 2)(3x^4y^3 - 2)$. Ans.

EXAMPLE 2. Factor $(a + b)^2 - 4m^2n^2$.

SOLUTION.—The square root of the first term is $a + b$. The square root of the last term is $2mn$.

Hence, $(a + b)^2 - 4m^2n^2 = (a + b + 2mn)(a + b - 2mn)$. Ans.

EXAMPLES FOR PRACTICE

Factor:

1. $a^2 - 16$ Ans. $(a + 4)(a - 4)$
2. $a^2 - 49c^2$ Ans. $(a + 7c)(a - 7c)$
3. $81x^4y^4 - 1$ Ans. $(9x^2y^2 + 1)(9x^2y^2 - 1)$
4. $(ax + by)^2 - 1$ Ans. $(ax + by + 1)(ax + by - 1)$
5. $25x^4y^2 - (bx + 1)^2$ Ans. $(5x^2y + bx + 1)(5x^2y - bx - 1)$
6. $(a + b)^2 - (a - b)^2$ Ans. $[(a + b) + (a - b)][(a + b) - (a - b)]$ $= (a + b + a - b)(a + b - a + b) = 2a \times 2b$ $= 4ab$

CASE IV

WWW.LIBRUM.ES

22. To Factor the Sum or the Difference of Two Perfect Cubes.

Let a and b represent any two numbers; then the sum and the difference of their cubes will be represented by $a^3 + b^3$ and $a^3 - b^3$, respectively. By division it can be shown that

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2$$
$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2$$

23. Rule.—*To factor the sum or the difference of two perfect cubes, extract the cube root of each term, connect these roots by the sign of the second term for one factor, and obtain the other factor by division.*

EXAMPLE.—Find the factors of $8x^6 - 27y^3$.

SOLUTION.—The cube root of the first term is $2x^2$. The cube root of the second term is $3y$. Since, the sign of the second term is $-$, the first factor is $2x^2 - 3y$. By division, the second factor is $4x^4 + 6x^2y + 9y^2$.

Therefore, $8x^6 - 27y^3 = (2x^2 - 3y)(4x^4 + 6x^2y + 9y^2)$. Ans.

EXAMPLES FOR PRACTICE

Factor:

1. $x^3 - y^3$ Ans. $(x-y)(x^2+xy+y^2)$
2. $m^3 + 64n^6$ Ans. $(m+4n^2)(m^2-4mn^2+16n^4)$
3. $27a^3 - 8x^3$ Ans. $(3a-2x)(9a^2+6ax+4x^2)$
4. $1 + 729m^{12}n^{15}$ Ans. $(1+9m^4n^5)(1-9m^4n^5+81m^8n^{10})$
5. $512a^3 - 64b^3$ Ans. $(8a-4b)(64a^2+32ab+16b^2)$

CASE V

24. To Factor Any Binomial.

Verify the following results by division:

$$\begin{cases}(x^3+y^3)\div(x+y) = x^2-xy+y^2\\(x^5+y^5)\div(x+y) = x^4-x^3y+x^2y^2-xy^3+y^4\end{cases}$$

$$\begin{cases}(x^3-y^3)\div(x-y) = x^2+xy+y^2\\(x^5-y^5)\div(x-y) = x^4+x^3y+x^2y^2+xy^3+y^4\end{cases}$$

Attention is directed to the following points in connection with these results:

I. *When the divisor is $x+y$, the terms of the quotient are alternately positive and negative.*

II. *When the divisor is $x-y$, the terms of the quotient are all positive.*

WWW.LIBRUM.US

III. *The number of terms in any of the quotients is equal to the exponent of x or y in the corresponding dividend.*

IV. *The exponent of x decreases by unity in each successive term of the quotient, while the exponent of y increases by unity in each successive term.*

25. It is frequently necessary, in mathemetical operations, to be able to distinguish between odd and even numbers. Let n denote any positive integer, odd or even; then, since two times any number is an even number, $2n$ will denote an even positive integer and $2n+1$ will denote an odd positive integer. With this notation x^{2n} means x with an even positive exponent and x^{2n+1} means x with an odd positive exponent.

26. It is now easy to prove that:

$x^{2n}-y^{2n}$ is always divisible by $x-y$ and by $x+y$. (I)

$x^{2n+1}-y^{2n+1}$ is always divisible by $x-y$ but never by $x+y$. (II)

$x^{2n+1}+y^{2n+1}$ is always divisible by $x+y$ but never by $x-y$. (III)

$x^{2n}+y^{2n}$ is divisible neither by $x+y$ nor by $x-y$. (IV)

By the remainder theorem, the remainder in dividing any polynomial by $x-y$ is equal to the result obtained by substituting y for x in the polynomial; and the remainder when any polynomial is divided by $x+y$ is equal to the result of substituting $-y$ for x in the polynomial. Hence, the remainder, when $x^{2n}-y^{2n}$ is divided by $x-y$, is $y^{2n}-y^{2n}=0$; and the remainder when $x^{2n}-y^{2n}$ is divided by $x+y$ is $(-y)^{2n}-y^{2n}$. Since any number, positive or negative, raised to an even power is positive, $(-y)^{2n}=y^{2n}$; therefore, $(-y)^{2n}-y^{2n}=y^{2n}-y^{2n}=0$. Consequently, as there is no remainder when $x^{2n}-y^{2n}$ is divided by either $x-y$ or $x+y$, $x^{2n}-y^{2n}$ is divisible by $x-y$ and also by $x+y$, which establishes theorem I.

Again, by the remainder theorem, the remainder in dividing $x^{2n+1}-y^{2n+1}$ by $x-y$ is $y^{2n+1}-y^{2n+1}=0$; and the remainder when $x^{2n+1}-y^{2n+1}$ is divided by $x+y$ is $(-y)^{2n+1}-y^{2n+1}$. Since an odd power of any negative number is negative, $(-y)^{2n+1}=-y^{2n+1}$. Therefore, the remainder $(-y)^{2n+1}-y^{2n+1}=-y^{2n+1}-y^{2n+1}=-2y^{2n+1}$. As there is no remainder when $x^{2n+1}-y^{2n+1}$ is divided by $x-y$, and there is a remainder when $x^{2n+1}-y^{2n+1}$ is divided by $x+y$, it follows that $x^{2n+1}-y^{2n+1}$ is divisible by $x-y$ but not by $x+y$, which establishes theorem II.

Similarly, the remainder obtained by dividing $x^{2n+1}+y^{2n+1}$ by $x-y$ is $y^{2n+1}+y^{2n+1}=2y^{2n+1}$; and the remainder obtained by dividing $x^{2n+1}+y^{2n+1}$ by $x+y$ is $(-y)^{2n+1}+y^{2n+1}=-y^{2n+1}+y^{2n+1}=0$. Therefore, $x^{2n+1}+y^{2n+1}$ is divisible by $x+y$ but not by $x-y$, which establishes theorem III.

The remainder when $x^{2n}+y^{2n}$ is divided by $x-y$ is $y^{2n}+y^{2n}=2y^{2n}$; and the remainder when $x^{2n}+y^{2n}$ is divided by $x+y$

is $(-y)^{2n}+y^{2n} = y^{2n}+y^{2n} = 2y^{2n}$. Therefore, $x^{2n}+y^{2n}$ is divisible neither by $x+y$ nor by $x-y$, and theorem IV is established.

NOTE.—The foregoing demonstration may be omitted until review.

27. The theorems of Art. **26**, when stated in words, are:

I. *The difference of the like even powers of two numbers is exactly divisible by the difference of the numbers and also by their sum, taken in the same order.*

II. *The difference of the like odd powers of two numbers is exactly divisible by the difference of the numbers, taken in the same order, but not by their sum.*

III. *The sum of the like odd powers of two numbers is exactly divisible by their sum but not by the difference of the numbers, taken in the same order.*

IV. *The sum of the like even powers of two numbers is* not *exactly divisible by either the sum or the difference of the two numbers.*

28. The easiest way to remember these results is to notice the simplest example in each of the four theorems and refer all other examples to it. Suppose it is required to know whether x^9-y^9 is divisible by $x-y$, or $x+y$, or by both. The exponent 9 is an odd number and the simplest example in this theorem is $x-y$, which is divisible by $x-y$, but not by $x+y$; therefore, it is inferred that x^9-y^9 is divisible by $x-y$ but not by $x+y$.

Again consider x^8-y^8. The exponent 8 is an even integer, and the simplest example in this case is x^2-y^2, which is divisible by both $x-y$ and $x+y$; hence, it is inferred that x^8-y^8 is divisible by both $x-y$ and $x+y$.

29. Rule.—*To resolve a number of the form x^n+y^n or of the form x^n-y^n into factors, determine by the theorems of Art.* **27**, *whether the number is divisible by*

$x+y$ or by $x-y$, and then obtain the other factor by division.

Example 1.—Resolve the number $216a^3+729b^3$ into factors.

Solution.—Since, $216a^3+729b^3 = (6a)^3+(9b)^3$, the given number is of the form x^3+y^3, where x is displaced by $6a$ and y by $9b$. Since, by theorem III, Art. **27**, x^3+y^3 is divisible by $x+y$, it follows that the given number is divisible by $6a+9b$. The quotient in dividing $216a^3+729b^3$ by $6a+9b$ is $36a^2-54ab+81b^2$.

Hence, $216a^3+729b^3 = (6a+9b)(36a^2-54ab+81b^2)$. Ans.

Example 2.—Resolve r^3-8 into two factors.

Solution.—Since, $r^3-8 = r^3-2^3$, the given number is of the form x^3-y^3, where x is displaced by r and y by 2. Since x^3-y^3 is divisible by $x-y$, it follows that r^3-8 is divisible by $r-2$. The quotient in dividing r^3-8 by $r-2$ is r^2+2r+4.

Hence, $r^3-8 = (r-2)(r^2+2r+4)$. Ans.

EXAMPLES FOR PRACTICE

Resolve into two factors:

1. $x^{11}-y^{11}$ Ans. $(x-y)(x^{10}+x^9y+x^8y^2+x^7y^3+x^6y^4+x^5y^5+x^4y^6+x^3y^7+x^2y^8+xy^9+y^{10})$
2. $27x^3+8y^3$ Ans. $(3x+2y)(9x^2-6xy+4y^2)$
3. $32x^5+243y^5$ Ans. $(2x+3y)(16x^4-24x^3y+36x^2y^2-54xy^3+81y^4)$
4. x^5-32 Ans. $(x-2)(x^4+2x^3+4x^2+8x+16)$
5. $729a^3-216b^3$ Ans. $(9a-6b)(81a^2+54ab+36b^2)$

WWW.LIBRUM.US

CASE VI

30. To Factor a Trinomial of the Form x^2+bx+c. If $x+4$ is multiplied by $x+3$, the product is $x^2+(4+3)x+12$, or $x^2+7x+12$.

Now, to factor $x^2+7x+12$, two factors of 12 must be found such that their algebraic sum is 7. By trying the different factors of 12, it is found that 3 and 4 are the factors whose sum is 7. The required factors are, therefore, $x+3$ and $x+4$.

Again if $x+2$ is multiplied by $x-3$, the product is $x^2+(2-3)x-6$, or x^2-x-6.

Now, to factor x^2-x-6 it is necessary to find two factors of -6, whose algebraic sum is -1, the coefficient of x in the given number. By trying the different factors of -6, it is found that $+2$ and -3 are the factors whose sum is -1. Hence, the required factors are $x-3$ and $x+2$.

In general, if $x+m$ is multiplied by $x+n$, the product is $x^2+(m+n)x+mn$. This product is formed by taking the sum of m and n for the coefficient of x in the second term, and by taking the product of m and n for the third term. Hence, x^2+bx+c can be factored if b is the algebraic sum of two factors of c.

31. Rule.—*To factor a number of the form x^2+bx+c, find two numbers which multiplied together give the third term, and which added together give the coefficient of x in the second term; x plus one number is one factor and x plus the other number is the other factor.*

EXAMPLE 1.—Factor $a^2+8a+12$.

SOLUTION.—The number $a^2+8a+12$ is of the form x^2+bx+c, where $a = x$, $b = 8$, and $c = 12$. Now find two numbers whose sum is 8 and whose product is 12. Since their product is positive they must both be positive or both negative, and since their sum is positive they must both be positive. The possible pairs of factors of 12 are 1 and 12, 2 and 6, 3 and 4. But since $2+6 = 8$, 2 and 6 are the factors to use.

Therefore, $a^2+8a+12 = (a+2)(a+6)$. Ans.

EXAMPLE 2.—Factor $m^2n^2-2mn-15$.

SOLUTION.—The number $m^2n^2-2mn-15$ is of the form x^2+bx+c, where $x = mn$, $b = -2$, and $c = -15$. Now find two numbers whose sum is -2 and whose product is -15. Since the product is negative, one factor must be negative and the other positive; and since the sum is negative, the negative factor must have the greater absolute value. The possible factors of -15 are 1 and -15, and 3 and -5. But since $3+(-5) = -2$, 3 and -5 are the factors to use.

Therefore, $m^2n^2-2mn-15 = (mn+3)(mn-5)$. Ans.

EXAMPLE 3.—Factor $x^2-5xy+6y^2$.

SOLUTION.—The number $x^2-5xy+6y^2$ is of the form x^2+bx+c, where $x = x$, $b = -5y$, and $c = 6y^2$. Now, find two numbers whose sum is $-5y$ and whose product is $6y^2$. Since the product is positive, the factors must both be positive or both negative; and since the sum is negative, both factors must be negative. The possible factors of $6y^2$ are $-y$ and $-6y$, and $-2y$ and $-3y$. But since $-2y+(-3y) = -5y$, $-2y$ and $-3y$ are the factors to use.

Therefore, $x^2-5xy+6y^2 = (x-2y)(x-3y)$. Ans.

EXAMPLES FOR PRACTICE

Factor the following:

1. $a^2 + 15a + 56$ Ans. $(a+7)(a+8)$
2. $b^2 - 4b - 21$ Ans. $(b+3)(b-7)$
3. $x^2 + 3x - 10$ Ans. $(x+5)(x-2)$
4. $y^2z^2 + 4yz - 5$ Ans. $(yz+5)(yz-1)$
5. $x^2 + 21x + 20$ Ans. $(x+1)(x+20)$
6. $(a+b)^2 - 3(a+b) - 18$ Ans. $(a+b-6)(a+b+3)$
7. $m^2 - 9mn + 20n^2$ Ans. $(m-5n)(m-4n)$
8. $a^2 - 4ax - 77x^2$ Ans. $(a+7x)(a-11x)$

CASE VII

32. To Factor a Trinomial of the Form $ax^2 + bx + c$.

If $2x+3$ is multiplied by $5x+1$, the product is $10x^2 + 17x + 3$. The part of the multiplication that gives the second term may be represented thus,

$$\begin{array}{c} 2x + 3 \\ \times \\ 5x + 1 \\ \hline (15+2)x \end{array}$$

The products of the terms connected by cross-lines are called **cross-products**, and the sum of the cross-products is the middle term of the trinomial product.

By inspection of the trinomial product $10x^2 + 17x + 3$, it is seen that:

1. The first term is the product of the first terms of the binomial factors.

2. The third term is the product of the second terms of the binomial factors.

3. The second term is the sum of the products of the first term of each binomial factor by the second term of the other.

Thus, to factor a trinomial of the form $ax^2 + bx + c$, it is necessary to determine two factors of the first term and two factors of the third term such that the algebraic sum of their cross-products is equal to the second term of the trinomial.

EXAMPLE.—Factor $2x^2 - x - 10$.

SOLUTION.—The factors of the first term $2x^2$ are $2x$, x. The factors of the third term -10 are $-1, 10$; $-10, 1$; $2, -5$; and $-2, 5$. In order